Prvi izpit iz Numerične matematike
9. junij 2023

1. **naloga:** Naj bo $D = \operatorname{diag}(d_1, \ldots, d_n)$ diagonalna matrika z $0 < d_1 < \ldots < d_n$ in naj bo $z = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^T \in \mathbb{R}^n$ vektor. Tvorimo matriko $M = \begin{bmatrix} D & z \end{bmatrix}$:

$$M = \begin{bmatrix} d_1 & & & & z_1 \\ & d_2 & & & z_2 \\ & & \ddots & & \vdots \\ & & & d_n & z_n \end{bmatrix}.$$

Izkaže se, da neničelne singularne vrednosti $\sigma_1, \ldots, \sigma_n$ matrike M zadoščajo nelinearni enačbi

$$1 + \sum_{i=1}^{n} \frac{z_i^2}{d_i^2 - w^2} = 0$$

in zadoščajo lastnosti prepletanja

$$0 < d_1 < \sigma_1 < d_2 < \sigma_2 < \ldots < d_n < \sigma_n < d_n + \|z\|.$$

Naj bo $n = 3, d_1 = 1, d_2 = 2, d_3 = 3$ in $z_1 = z_2 = z_3 = 1$.

Naloga: Z enim korakom tangentne metode s smiselno izbranima začetnima približkoma ocenite najmanjšo in največjo singularno vrednost σ_1, σ_3.

Rešitev. Za uporabo tangentne metode potrebujemo odvod funkcije

$$f(w) := 1 + \frac{1}{1 - w^2} + \frac{1}{4 - w^2} + \frac{1}{9 - w^2},$$

t.j.,

$$f'(w) = \frac{2w}{(1 - w^2)^2} + \frac{2w}{(4 - w^2)^2} + \frac{2w}{(9 - w^2)^2}.$$

Korak tangentne metode je

$$w_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)}.$$

Po lastnosti prepletanja velja

$$1 < \sigma_1 < 2 < \sigma_2 < 3 < \sigma_3 < 3 + \sqrt{3}.$$

Za oceno σ_1 je smiselno uporabiti $w_0 = \frac{3}{2}$. Ker je $f(1.5) = 0.920$ in $f'(1.5) = 2.965$, sledi

$$w_1 \approx 1.5 - \frac{0.920}{2.965} = 1.190.$$

Za oceno σ_3 je smiselno uporabiti $w_0 = 3.5$. Ker je $f(3.5) = 0.482$ in $f'(3.5) = 0.821$, sledi

$$w_1 \approx 3.5 - \frac{0.482}{0.821} = 2.913.$$

2. **naloga:** Naj bo f integrabilna funkcija na $[0,1]$, katere integral želimo izračunati po formuli

$$\int_0^1 f(x)dx \approx \alpha f(0) + \beta f\Big(\frac{2}{3}\Big) + \gamma f(1).$$

(a) Določite $\alpha, \beta, \gamma \in \mathbb{R}$, da bo formula čim višjega reda.

(b) Izračunajte integral $\int_0^2 e^{x^2} dx$ z dvakratno uporabo zgornje formule s korakom $h = 1$.

(c) Izračunajte integral $\int_0^2 e^{x^2} dx$ s sestavljenim trapeznim pravilom s korakom $h = 1$.

Rešitev.

(a) Ker imamo v metodi 3 proste parametre, jih lahko izberemo tako, da bo metoda reda 2. Za f zaporedoma vstavimo $1, x, x^2$ in dobimo po metodi nedoločenih koeficientov linearni sistem

$$\int_0^1 1dx = [x]_0^1 = 1 = \alpha + \beta + \gamma, \tag{1}$$

$$\int_0^1 xdx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} = \beta\frac{2}{3} + \gamma, \tag{2}$$

$$\int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} = \beta\frac{4}{9} + \gamma. \tag{3}$$

Iz (2)–(3) sledi $\frac{2}{9}\beta = \frac{1}{6}$ in zato $\beta = \frac{3}{4}$. Od tod pa sledi $\gamma = 0$ in $\alpha = \frac{1}{4}$.

(b) Velja

$$\begin{aligned}\int_0^2 e^{x^2}dx &= \int_0^1 e^{x^2}dx + \int_1^2 e^{x^2}dx = \int_0^1 e^{x^2}dx + \int_0^1 e^{(\tilde{x}+1)^2}d\tilde{x} \\ &= \frac{1}{4}e^{0^2} + \frac{3}{4}e^{(\frac{2}{3})^2} + \frac{1}{4}e^{1^2} + \frac{3}{4}e^{(\frac{5}{3})^2} \\ &\approx 14.162,\end{aligned}$$

kjer smo v drugi enakosti naredili substitucijo $x = \tilde{x} + 1$, $dx = d\tilde{x}$.

(c) Sestavljeno trapezno pravilo za $f(x) = e^{x^2}$ na intervalu $[0,2]$ s korakom $h = 1$ je enako

$$\int_0^2 e^{x^2}dx = \frac{1}{2}(e^{0^2} + 2e^{1^2} + e^{2^2}) \approx 30.52.$$

3. **naloga:** Naj bo $f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$, $u_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $u_2 \in [0, 2\pi)$ parametrizacija torusa:

$$f(u_1, u_2) = \Big(\big(2 + \cos(u_1)\big)\cos(u_2), \big(2 + \cos(u_1)\big)\sin(u_2), \sin(u_1)\Big).$$

(a) Matrika G, ki se imenuje metrični tenzor, je definirana kot:

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_1} & \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_2} \\ \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_2}\frac{\partial f_k}{\partial u_1} & \sum_{k=1}^{3} \frac{\partial f_k}{\partial u_2}\frac{\partial f_k}{\partial u_2} \end{bmatrix}.$$

Po kratkem računu sledi, da je $g_{12} = g_{21} = 0$, $g_{22} = (2 + \cos(u_1))^2$. Izračunajte še g_{11} in inverz matrike G:

$$G^{-1} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}.$$

(b) Za $i, j, k = 1, 2$ so Christoffelovi simboli Γ_{ij}^k definirani kot

$$\Gamma_{ij}^k = \sum_{\ell=1}^{2} \left\langle \begin{bmatrix} \frac{\partial^2 f_1}{\partial u_i \partial u_j} \\ \frac{\partial^2 f_2}{\partial u_i \partial u_j} \\ \frac{\partial^2 f_3}{\partial u_i \partial u_j} \end{bmatrix}, \begin{bmatrix} \frac{\partial f_1}{\partial u_\ell} \\ \frac{\partial f_2}{\partial u_\ell} \\ \frac{\partial f_3}{\partial u_\ell} \end{bmatrix} \right\rangle \cdot h_{\ell k},$$

kjer $\langle \cdot, \cdot \rangle$ označuje običajen skalarni produkt vektorjev. Po kratkih računih sledi

$$\Gamma_{11}^2 = 0, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = -\frac{\sin(u_1)}{2 + \cos(u_1)},$$
$$\Gamma_{22}^1 = (2 + \cos(u_1))\sin(u_2), \quad \Gamma_{22}^2 = 0.$$

Izračunajte še Γ_{11}^1.

(c) Najkrajše poti na torusu $\gamma(t) = (u_1(t), u_2(t))$ zadoščajo naslednjima dvema diferencialnima enačbama drugega reda:

$$\begin{aligned} \frac{d^2 u_1}{dt^2} + \sum_{i,j=1}^{2} \Gamma_{ij}^1 \frac{du_i}{dt}\frac{du_j}{dt} &= 0, \\ \frac{d^2 u_2}{dt^2} + \sum_{i,j=1}^{2} \Gamma_{ij}^2 \frac{du_i}{dt}\frac{du_j}{dt} &= 0. \end{aligned} \tag{4}$$

Napišite eksplicitno obliko sistema (4).

(d) Prevedite sistem (3c) na sistem prvega reda z uvedbo novih spremenljivk

$$x_1(t) = u_1(t),\ x_2(t) = \frac{du_1}{dt},\ x_3(t) = u_2(t),\ x_4(t) = \frac{du_2}{dt}.$$

(e) Za začetno točko $(u_1(0), u_2(0)) = (0, 0)$ in odvoda $\left(\frac{du_1}{dt}(0), \frac{du_2}{dt}(0)\right) = (1, 1)$ ocenite $x_i(0.1)$, $i = 1, 2, 3, 4$, z enim korakom Eulerjeve methode na sistemu iz (3d).

Rešitev. Velja

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \\ \frac{\partial f_3}{\partial u_1} \end{bmatrix} = \begin{bmatrix} -\sin(u_1)\cos(u_2) \\ -\sin(u_1)\sin(u_2) \\ \cos(u_1) \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_2} \end{bmatrix} = \begin{bmatrix} -(2+\cos(u_1))\sin(u_2) \\ (2+\cos(u_1)\cos(u_2) \\ 0 \end{bmatrix}.$$

(a) Velja:

$$\begin{aligned}
\sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_1} &= \sin^2(u_1)\cos^2(u_2) + \sin^2(u_1)\sin^2(u_2) + \cos^2(u_1) \\
&= \sin^2(u_1)\Big(\cos^2(u_2) + \sin^2(u_2)\Big) + \cos^2(u_1) = \sin^2(u_1) + \cos^2(u_1) = 1, \\
\sum_{k=1}^{3} \frac{\partial f_k}{\partial u_1}\frac{\partial f_k}{\partial u_2} &= \sin(u_1)\cos(u_2)(2+\cos(u_1))\sin(u_2) \\
&\qquad - \sin(u_1)\sin(u_2)(2+\cos(u_1))\cos(u_2) = 0, \\
\sum_{k=1}^{3} \frac{\partial f_k}{\partial u_2}\frac{\partial f_k}{\partial u_2} &= (2+\cos(u_1))^2\sin(u_2)^2 + (2+\cos(u_1))^2\cos(u_2)^2 \\
&= (2+\cos(u_1))^2(\sin^2(u_2)+\cos^2(u_2)) = (2+\cos(u_1))^2.
\end{aligned}$$

Torej je

$$G = \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(u_1))^2 \end{bmatrix} \quad \text{in} \quad G^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & (2+\cos(u_1))^{-2} \end{bmatrix}.$$

(b) Velja

$$\begin{bmatrix} \frac{\partial^2 f_1}{\partial u_1 \partial u_1} \\ \frac{\partial^2 f_2}{\partial u_1 \partial u_1} \\ \frac{\partial^2 f_3}{\partial u_1 \partial u_1} \end{bmatrix} = \begin{bmatrix} -\cos(u_1)\cos(u_2) \\ -\cos(u_1)\sin(u_2) \\ -\sin(u_1) \end{bmatrix}.$$

Torej je

$$\begin{aligned}
\Gamma_{11}^{1} &= \sum_{\ell=1}^{2} \left\langle \begin{bmatrix} \frac{\partial^2 f_1}{\partial u_1 \partial u_1} \\ \frac{\partial^2 f_2}{\partial u_1 \partial u_1} \\ \frac{\partial^2 f_3}{\partial u_1 \partial u_1} \end{bmatrix}, \begin{bmatrix} \frac{\partial f_1}{\partial u_\ell} \\ \frac{\partial f_2}{\partial u_\ell} \\ \frac{\partial f_3}{\partial u_\ell} \end{bmatrix} \right\rangle h_{\ell 1} = \left\langle \begin{bmatrix} \frac{\partial^2 f_1}{\partial u_1 \partial u_1} \\ \frac{\partial^2 f_2}{\partial u_1 \partial u_1} \\ \frac{\partial^2 f_3}{\partial u_1 \partial u_1} \end{bmatrix}, \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \\ \frac{\partial f_3}{\partial u_1} \end{bmatrix} \right\rangle h_{11} \\
&= \cos(u_1)\sin(u_1)\cos^2(u_2) + \cos(u_1)\sin(u_1)\sin^2(u_2) - \sin(u_1)\cos(u_1) \\
&= \cos(u_1)\sin(u_1)\Big(\cos^2(u_2) + \sin^2(u_2)) - \sin(u_1)\cos(u_1) \\
&= \cos(u_1)\sin(u_1) - \cos(u_1)\sin(u_1) = 0.
\end{aligned}$$

(c) Sistem je enak

$$\begin{aligned} \frac{d^2u_1}{dt^2} + (2+\cos(u_1))\sin(u_2)\frac{du_2}{dt}\frac{du_2}{dt} &= 0, \\ \frac{d^2u_2}{dt^2} - 2\frac{\sin(u_1)}{2+\cos(u_1)}\frac{du_1}{dt}\frac{du_2}{dt} &= 0. \end{aligned} \tag{5}$$

(d) Pripadajoč sistem prvega reda sistema (5) je

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \\ \frac{dx_2}{dt} &= -(2+\cos(x_1))\sin(x_3)x_4^2, \\ \frac{dx_3}{dt} &= x_4, \\ \frac{dx_4}{dt} &= \frac{2\sin(x_1)}{2+\cos(x_1)}x_2x_4. \end{aligned} \tag{6}$$

(e) Z uporabo Eulerjeve metode na sistemu (6) z $t_0 = 0$ in $h = 0.1$ dobimo

$$\begin{aligned} x_1(0.1) &= x_1(0) + 0.1 \cdot x_2(0) \\ &= 0 + 0.1 \cdot 1 = 0.1, \\ x_2(0.1) &= x_2(0) - 0.1 \cdot (2+\cos(x_1(0)))\sin(x_3(0))x_4^2(0) \\ &= 1 - 0.1 \cdot (2+\cos(0))\sin(0)1^2 = 1, \\ x_3(0.1) &= x_3(0) + 0.1 \cdot x_4(0) \\ &= 0 + 0.1 \cdot 1 = 0.1, \\ x_4(0.1) &= x_4(0) + 0.1 \cdot \frac{2\sin(x_1(0))}{2+\cos(x_1(0))}x_2(0)x_4(0) \\ &= 1 + 0.1 \cdot \frac{2\sin(0)}{2+\cos(0)} = 1. \end{aligned}$$