Computational topology
Lab work 10

1. Two different monotonic functions are given on the simplicial complex X:

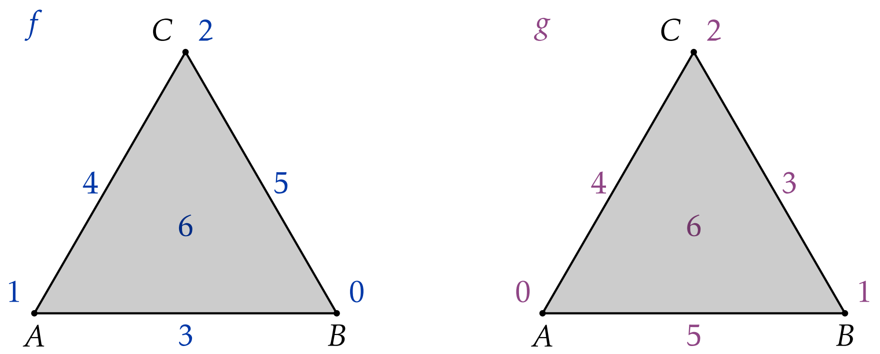
$$\begin{aligned} f &= \{(A,1),(B,0),(C,2),(AB,3),(AC,4),(BC,5),(ABC,6)\}, \\ g &= \{(A,0),(B,1),(C,2),(AB,5),(AC,4),(BC,3),(ABC,6)\}. \end{aligned}$$

   (a) Create the corresponding filtrations of subcomplexes.

   (b) Draw the barcode diagrams and the persistence diagrams in dimensions 0 and 1.

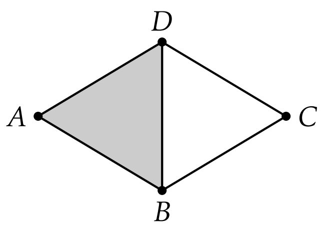
   (c) Construct the boundary matrices D_f and D_g from the two filtrations.

   (d) Use the matrix reduction to compute persistence.




```
for j = 1 to m:
    while there exists j0 < j with low(j0) = low(j):
        D[:, j] = D[:, j] - D[low(j), j]/D[low(j), j0] * D[:, j0]
```

2. Let K be the simplicial complex drawn below.



A filtration on K is given as

- $K_1 = \{A, C\}$,
- $K_2 = K_1 \cup \{B, D, BD\}$,
- $K_3 = K_2 \cup \{AD, BC\}$,
- $K_4 = K_3 \cup \{CD\}$,
- $K_5 = K_4 \cup \{AB\}$,
- $K_6 = K_5 \cup \{ABD\}$.

   (a) Draw the barcode diagrams and the persistence diagrams in dimensions 0 and 1.

   (b) Construct the boundary matrices D of this filtration.

   (c) Column–reduce D to compute persistence.

For a simplicial complex K a **filtration** of K is a sequence of simplicial complexes

$$K_0 \leq K_1 \leq \ldots \leq K_n = K.$$

We can obtain filtrations from the skeleta of K (filtered by dimension), the Vietoris-Rips or Cech complexes (filtered by the radius), from discrete Morse functions, etc. A **critical event** in a filtration is when the homotopy type of K_{i+1} is different from the homotopy type of K_i.

Given a filtration

$$K_0 \leq K_1 \leq \ldots \leq K_n = K,$$

the **persistent homology group** H_p^{jt} of the pair (j,t) is the p^{th} homology group of the complex K_j, computed in K_t:

$$H_p^{jt} = \frac{Z_p(K_j)}{B_p(K_t) \cap Z_p(K_j)}.$$

The corresponding **persistent Betti number** is $b_p^{jt} = \text{rank} H_p^{jt}$ and is equal to the number of independent non-trivial homology classes of $H_p(K_j)$ that are still non-trivial in $H_p(K_t)$. A homology class $\gamma \in H_p(K_i)$ **is born** in K_i, if $\gamma \notin H_p^{i-1,j}$. A homology class that was born in K_i, **dies** in K_j for some $j > i$, if it is non-trivial in $K_{i+1}, \ldots, K_{j-1}$ and becomes a boundary in K_j. The **persistence** of the class γ is the interval between the birth and death of γ: $\text{pers}(\gamma) = [i, j)$. If γ is born and never dies then its persistence is $[i, \infty)$ and γ is a generator of the homology of K.

A **barcode** is a diagram that contains for each class γ with persistence $[i,j)$ an interval (a bar) from i to j.
A **persistence diagram** is obtained by associating to each class γ with persistence $[i,j)$ a point in the plane with coordinates (i,j). If the multiplicity

$$\mu_p^{ij} = (b_p^{i,j-1} - b_p^{ij}) - (b_p^{i-1,j-1} - b_p^{i-1,j})$$

is greater than 1, we add it as a label to the point (i,j). If the persistence of γ is $[i,\infty)$, we draw a vertical ray starting at (i,i). Finally, we also include the diagonal

$$\Delta = \{(x,x) \mid x \in \mathbb{R}\}.$$

Given a simplicial complex $K = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$, we can define a matrix D by

$$D_{i,j} = \begin{cases} 1; & \sigma_i < \sigma_j, \\ 0; & \text{otherwise.} \end{cases}$$

Define

$$\text{low(j)} = \text{row index of the lowest 1 in column } j.$$

We use column operations to reduce D to R. The matrix R is reduced if $\text{low}(j) \neq \text{low}(j_0)$ for all $j \neq j_0$. The algorithm for obtaining R from D is:

```
R = D
for j = 1 to m:
    while there exists j_0 < j with low(j_0) = low(j):
        add column R[:, j_0] to column R[:, j]
```