# Mathematical Modelling Exam

June 29th, 2023

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let $A \in \mathbb{R}^{n\times m}$ be a given matrix and $X \in \mathbb{R}^{m\times n}$ a matrix satisfying the following property:

$$\text{If the sytem } Ax = b \text{ is solvable, then } x = Xb \text{ is one of the solutions.} \qquad (1)$$

   (a) Show that $X$ is a generalized inverse of $A$.

   (b) Give an example of a matrix $A \in \mathbb{R}^{2\times 2}$ and a matrix $X$ such that (1) is satisfied, but $X$ is not the Moore-Penrose inverse of $A$.

2. Let $f(x, y, z) = \big(2 - \sqrt{x^2 + y^2}\big)^2 + z^2 - 1$ be a function of three variables.

   (a) Check that every point on the torus with the center of the hole being the origin, the distance from the origin to the center of the torus tube being 2 and the radius of the tube being 1, is a solution of the equation $f(x, y, z) = 0$.

   (b) Perform one step of Gauss-Newton method to approximate the point on the torus $f(x, y, z) = 0$ given the initial approximation $(x_0, y_0, z_0) = \big(1, 1, 1\big)$.

3. Let

$$f(u, v) = (u, v, u^2 - v^2), \quad (u, v) \in \mathbb{R}^2$$

   be the parametric surface $\Pi$.

   (a) Sketch few coordinate curves for each parameter $u$ and $v$.

   (b) Sketch the surface $\Pi$.

   (c) Determine the parametric equation of the tangent plane at every point of $\Pi$.

   (d) Write the expression for the length of the curve $\gamma(t) = (t, t^2, t^2 - t^4)$ on $\Pi$ between the points $(0, 0, 0)$ and $(1, 1, 0)$ and approximate it with a simple trapezoid rule.

4. Let

$$y' + 2y = 2 - e^{-4x}$$

   be the differential equation (DE) with the initial condition $y(0) = 1$.

   (a) Solve the DE exactly.

   (b) Use Euler's Method with a step size $h = 0.1$ to find approximate values of the solution at $x = 0.1$ and $x = 0.2$. Compare these approximations with the exact values of the solution at these points.