Mathematical Modelling Exam

June 16th, 2024

You have 90 minutes to solve the problems. The numbers in $[\cdot]$ represent points.

1. Solve the following tasks:

   (a) **[6]** Let $A \in \mathbb{R}^{n\times m}$ be a matrix and $G \in \mathbb{R}^{m\times n}$ one of its generalized inverses. Check that
   $$\ker A = \{(GA - I)z \colon z \in \mathbb{R}^m\}.$$

   (b) **[6]** Let $\mathcal{C}$ be a circle in the xz–plane with radius r, centered at $(R, 0)$, $R > r$. Check that a surface obtained by revolving $\mathcal{C}$ around the z–axis satisfies the following cartesian equation:
   $$(x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2(x^2 + y^2).$$

2. Given are points $(0, 4)$, $(1, 4)$, $(2, 2)$, $(3, 6)$. We would like to approximate the points in terms of the least squares error method with the function of the form
   $$g(x) = C(x - 2) + D(x - 2)^2$$

   (a) **[2]** Write down the linear system for unknowns C, D.

   (b) **[8]** Compute the Moore–Penrose inverse A^+ of the matrix A of this system.

   (c) **[2]** Solve the system using A^+.

3. Let
   $$\mathbf{r}(t) = (t^5 - 4t^3, t^2)$$
   be a parametrisation of a curve $\mathcal{C}$ in $\mathbb{R}^2$.

   (a) **[8]** Sketch the curve (determine intersections with both axes, self-intersections, horizontal and vertical tangents).

   (b) **[5]** The curve $\mathcal{C}$ has one loop. Compute the area of the region inside the loop.
   Hint: When using the area formula determined by $\mathbf{r}(t)$ be careful on the sign change of the integrand.

4. Let
   $$y'' - y'y^2 + y = 0,\ y(0) = 1,\ y'(0) = 0$$
   be a second order differential equation (DE).

   (a) **[3]** Translate the DE into a system of first order DEs.

   (b) **[10]** Use the Runge–Kutta method with a Butcher tableau
   $$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$
   and a step size $h = 0.2$ to estimate $y(0.2)$.