Mathematical Modelling Exam

June 16th, 2024

You have 90 minutes to solve the problems. The numbers in $[\cdot]$ represent points.

1. Solve the following tasks.

   (a) **[6]** Let $A \in \mathbb{R}^{n\times m}$ be a matrix and $G \in \mathbb{R}^{m\times n}$ one of its generalized inverses. Check that

   $$\ker A = \{(GA - I)z \colon z \in \mathbb{R}^m\}.$$

   Solution. $(\subseteq)$: Let $v \in \ker A$. We have to check that there is $z \in \mathbb{R}^m$ such that $v = (GA - I)z$. By $(GA - I)(-v) = -GAv + v = v$, a good choice for z is $-v$. $(\supseteq)$: Let $z \in \mathbb{R}^m$. Then $A(GA - I)z = AGAz - Az = Az - Az = 0$, where we used that G is a generalized inverse of A in the second equality.

   (b) **[6]** Let $\mathcal{C}$ be a circle in the xz–plane with radius r, centered at $(R, 0)$, $R > r$. Check that a surface obtained by revolving $\mathcal{C}$ around the z–axis satisfies the following cartesian equation:

   $$(x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2(x^2 + y^2).$$

   Solution. The surface is a torus with a parametrization

   $$\begin{aligned} x(\varphi, \phi) &= (R + r\cos\phi)\cos\varphi, \\ y(\varphi, \phi) &= (R + r\cos\phi)\sin\varphi, \\ z(\varphi, \phi) &= r\sin\phi, \end{aligned}$$

   $\varphi \in [0, 2\pi)$, $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence,

   $$\begin{aligned} &(x^2 + y^2 + z^2 + R^2 - r^2)^2 = \\ &= \left((R + r\cos\phi)^2\cos^2\varphi + (R + r\cos\phi)^2\sin^2\varphi + r^2\sin^2\phi + R^2 - r^2\right)^2 \\ &= \left((R + r\cos\phi)^2 + r^2\sin^2\phi + R^2 - r^2\right)^2 \\ &= (R^2 + 2Rr\cos\phi + r^2\cos^2\phi + r^2\sin^2\phi + R^2 - r^2)^2 \\ &= (2R^2 + 2Rr\cos\phi)^2 = 4R^2(R + r\cos\phi)^2 = 4R^2(x^2 + y^2). \end{aligned}$$

2. Given are points $(0, 4)$, $(1, 4)$, $(2, 2)$, $(3, 6)$. We would like to approximate the points in terms of the least squares error method with the function of the form

   $$g(x) = C(x - 2) + D(x - 2)^2$$

   (a) **[2]** Write down the linear system for unknowns C, D.

   Solution.

   $$A\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} (0-2) & (0-2)^2 \\ (1-2) & (1-2)^2 \\ (2-2) & (2-2)^2 \\ (3-2) & (3-2)^2 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 6 \end{pmatrix}.$$

(b) **[8]** Compute the Moore–Penrose inverse A^+ of the matrix A of this system.

Solution.

$$\det(A^TA-\lambda I)=\det\begin{pmatrix}6-\lambda & -8\\ -8 & 18-\lambda\end{pmatrix}=\lambda^2-24\lambda+44=(\lambda-2)(\lambda-22).$$

$$\ker(A^TA-2I)=\ker\begin{pmatrix}4 & -8\\ -8 & 16\end{pmatrix}=\ker\begin{pmatrix}1 & -2\\ 0 & 0\end{pmatrix}=\mathrm{Lin}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\},$$
$$\ker(A^TA-22I)=\ker\begin{pmatrix}-16 & -8\\ -8 & -4\end{pmatrix}=\ker\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix}=\mathrm{Lin}\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\},$$

Hence, $\sigma_1=\sqrt{22}$, $\sigma_2=\sqrt{2}$, $v_1=\frac{1}{\sqrt5}\begin{pmatrix}1\\-2\end{pmatrix}$ $v_2=\frac{1}{\sqrt5}\begin{pmatrix}2\\1\end{pmatrix}$ and

$$u_1=\frac{Av_1}{\sigma_1}=\frac{1}{\sqrt{110}}\begin{pmatrix}-10\\-3\\0\\-1\end{pmatrix},\qquad u_2=\frac{Av_2}{\sigma_1}=\frac{1}{\sqrt{10}}\begin{pmatrix}0\\-1\\0\\3\end{pmatrix}.$$

Finally,

$$A^+=\sigma_1^{-1}v_1u_1^T+\sigma_2^{-1}v_2u_2^T=\frac{1}{110}\begin{pmatrix}-10 & -3 & 0 & -1\\ 20 & 6 & 0 & 2\end{pmatrix}+\frac{1}{10}\begin{pmatrix}0 & -2 & 0 & 6\\ 0 & -1 & 0 & 3\end{pmatrix}$$
$$=\begin{pmatrix}-\frac{1}{11} & -\frac{5}{22} & 0 & \frac{13}{22}\\ \frac{2}{11} & -\frac{1}{22} & 0 & \frac{7}{22}\end{pmatrix}$$

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(c) **[2]** Solve the system using A^+.

Solution.

$$\begin{pmatrix}C\\D\end{pmatrix}=A^+b=\begin{pmatrix}\frac{25}{11}\\ \frac{27}{11}\end{pmatrix}.$$

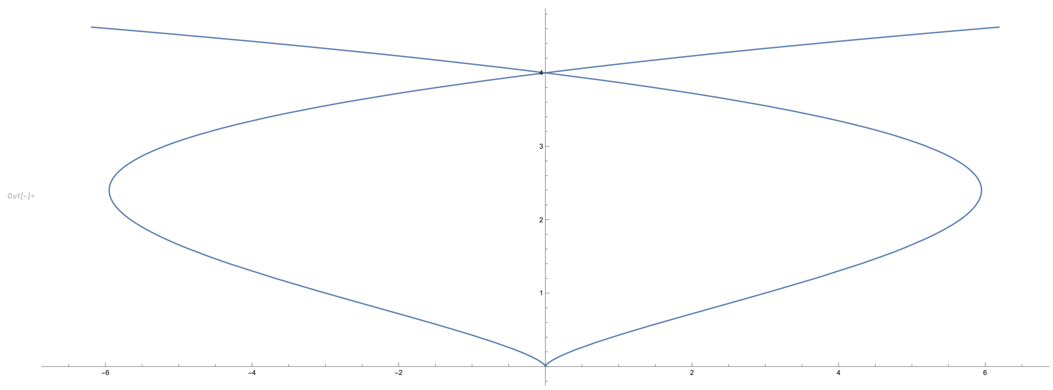
3. Let

$$\mathbf{r}(t)=(t^5-4t^3,t^2)$$

be a curve $\mathcal{C}$ in $\mathbb{R}^3$.

(a) **[8]** Sketch the curve (determine intersections with both axes, self-intersections, horizontal and vertical tangents).

Solution.



We have:

i. $\lim_{t\to-\infty} \mathbf{r}(t) = \begin{pmatrix} -\infty \\ \infty \end{pmatrix}$, $\quad \lim_{t\to\infty} \mathbf{r}(t) = \begin{pmatrix} \infty \\ \infty \end{pmatrix}$.

ii. $\mathbf{r}(t_1) = \mathbf{r}(t_2) \ \Leftrightarrow\ \begin{pmatrix} t_1^5 - 4t_1^3, \\ t_1^2 \end{pmatrix} = \begin{pmatrix} t_2^5 - 4t_2^3, \\ t_2^2 \end{pmatrix}$. From $t_1^2 = t_2^2$ it follows that $t_2 = -t_1$ ($t_1 = t_2$ is clearly not interesting). Then $t_1^5 - 4t_1^3 = (-t_1)^5 - 4(-t_1)^3 \ \Leftrightarrow\ 2(t_1^5 - 4t_1^3) = 0 \ \Leftrightarrow\ t_1 \in \{0, 2, -2\}$. So $(0, 4)$ is a self-intersection (for $t_1 = 2$, $t_2 = -2$).

iii. $x(t) = 0 \ \Leftrightarrow\ t^5 - 4t^3 = 0 \ \Leftrightarrow\ t^3(t^2 - 4) = 0 \ \Leftrightarrow\ t \in \{0, -2, 2\}$. So intersections with the y axis are $(0, 4)$, $(0, 0)$.

iv. $y(t) = 0 \ \Leftrightarrow\ t^2 = 0 \ \Leftrightarrow\ t = 0$. So the intersection with the x axis is $(0, 0)$.

v. $x'(t) = 0 \ \Leftrightarrow\ 5t^4 - 12t^2 = 0 \ \Leftrightarrow\ t^2(5t^2 - 12) = 0 \ \Leftrightarrow\ t \in \{0, \frac{2\sqrt{3}}{5}, -\frac{2\sqrt{3}}{5}\}$. So candidates for vertical tangents are $(0, 0)$, $(-1.17, \frac{12}{25})$, $(1.17, \frac{12}{25})$.

vi. $y'(t) = 0 \ \Leftrightarrow\ 2t = 0 \ \Leftrightarrow\ t = 0$. So the candidate for the horizontal tangent is in the point $(0, 0)$.

vii. In the points $(-1.17, \frac{12}{25})$, $(1.17, \frac{12}{25})$ there is really a vertical tangent, since those points are not singularities. The point $(0, 0)$ is a singularity and hence there is no tangent.

(b) **[5]** The curve $\mathcal{C}$ has one loop. Compute the area of the region inside the loop. **Hint:** When using the area formula determined by $\mathbf{r}(t)$ be careful on the sign change of the integrand.

Solution.
The area A inside the loop is the area $\mathbf{r}(t)$ describes for $t \in [-2, 2]$:

$$\begin{aligned} A &= \frac{1}{2}\int_{-2}^{2} |x(t)y'(t) - x'(t)y(t)| dt = \frac{1}{2}\int_{-2}^{2} |2(t^5 - 4t^3)t - (5t^4 - 12t^2)t^2| dt \\ &= \frac{1}{2}\int_{-2}^{2} t^4|-4 + 3t^2| dt = \int_{0}^{2} t^4|-4 + 3t^2| dt \\ &= -\int_{0}^{\frac{2}{\sqrt{3}}} t^4(-4 + 3t^2) dt + \int_{\frac{2}{\sqrt{3}}}^{2} t^4(-4 + 3t^2) dt \\ &= -\Big[-\frac{4}{5}t^5 + \frac{3}{7}t^7\Big]_0^{2/\sqrt{3}} + \Big[-\frac{4}{5}t^5 + \frac{3}{7}t^7\Big]_{2/\sqrt{3}}^{2} \approx 30.2 \end{aligned}$$

4. Let

$$y'' - y'y^2 + y = 0,\ y(0) = 1,\ y'(0) = 0$$

be a second order differential equation (DE).

(a) **[3]** Translate the DE into a system of first order DEs.

Solution. We introduce new variables $y_1 = y$, $y_2 = y'$. Hence, $y_2' - y_1' y_1^2 + y_1 = 0$ and the system becomes

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_2 y_1^2 - y_1 \end{pmatrix} =: \begin{pmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) **[10]** Use the Runge–Kutta method with a Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

and a step size $h = 0.2$ to estimate $y(0.2)$.

Solution. The RK method is

$$\begin{aligned} \vec{y}(x+h) &= \vec{y}(x) + \frac{1}{2}(\vec{k}_1 + \vec{k}_2), \\ \vec{k}_1 &= h\vec{f}(x, \vec{y}(x)), \\ \vec{k}_2 &= h\vec{f}(x+h, \vec{y}(x) + \vec{k}_1). \end{aligned}$$

So

$$\vec{y}(0.2) = \begin{pmatrix} y_1(0.2) \\ y_2(0.2) \end{pmatrix} = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} + \frac{1}{2}\Big(0.2\vec{f}(0, \vec{y}(0)) + 0.2\vec{f}(0.2, \vec{y}(0) + \vec{k}_1)\Big).$$

Further on,

$$\begin{aligned} \vec{k}_1 &= 0.2\vec{f}(0, \vec{y}(0)) = 0.2 \begin{pmatrix} f_1(0,1,0) \\ f_2(0,1,0) \end{pmatrix} = 0.2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.2 \end{pmatrix}, \\ \vec{k}_2 &= 0.2\vec{f}(0.2, \vec{y}(0) + \vec{k}_1) = 0.2\vec{f}(0.2, \begin{pmatrix} 1 \\ -0.2 \end{pmatrix}) = 0.2 \begin{pmatrix} f_1(0.2, 1, -0.2) \\ f_2(0.2, 1, -0.2) \end{pmatrix} \\ &= 0.2 \begin{pmatrix} -0.2 \\ -0.2 - 1 \end{pmatrix} = \begin{pmatrix} -0.04 \\ -0.24 \end{pmatrix}, \\ \vec{y}(0.2) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 0 \\ -0.2 \end{pmatrix} + \begin{pmatrix} -0.04 \\ -0.24 \end{pmatrix} \right) = \begin{pmatrix} 0.98 \\ -0.22 \end{pmatrix}. \end{aligned}$$

So, $y(0.2) \approx 0.98$.