Mathematical Modelling Exam

June 28th, 2022

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Let
$$A = \begin{bmatrix} 0 & 2 \\ -2 & -1 \\ -2 & 0 \end{bmatrix}$$
be a matrix.
   (a) Compute the truncated singular value decomposition of A.
   (b) Does there exist a matrix $B \in \mathbb{R}^{3\times 2}$ of rank 1 such that $\|A - B\|_F = 1$? If yes, compute it, otherwise justify, why it does not exist.

2. Let $f(x, y, z) = x^2 + 3xy + yz^3$ be a function of three variables.
   (a) Compute the gradient ∇f.
   (b) Perform one step of Newton's method to approximate the stationary point of f using the initial approximation $(x_0, y_0, z_0) = \left(1, 0, \frac{1}{\sqrt{3}}\right)$.

3. Let
$$f(t) = (t^3 - 5t^2 + 3t + 11, t^2 - 2t + 3), \quad t \in \mathbb{R}$$
be the parametric curve.
   (a) Find all points on the curve, where the tangent is horizontal or vertical.
   (b) Find all self-intersections.
   (c) Sketch the curve.

4. Let
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$
where $A \in \mathbb{R}^{3\times 3}$, be a system of differential equations with the following three solutions:
$$e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$
   (a) Write down a general solution of the system.
   (b) Determine the matrix A.
   (c) Write down a third order differential equation with constants coefficients, which is transformed into the above system.