Mathematical Modelling Exam

17. 6. 2020

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 105 minutes to solve the problems.

1. (a) Prove that if AA^T is an invertible matrix, then $A^\mathsf{T}(AA^\mathsf{T})^{-1}$ is the Moore–Penrose inverse A^+ of the matrix A (i.e. check that it satifies all four requirements for A^+).

   (b) Find the point on the intersection of the planes $x + y + z = 0$ and $x - y = 1$ that is closest to the origin:

      i. write down the matrix of the system for the intersection and find its Moore-Penrose inverse,

      ii. among all solutions of the system find the one closest to the origin.

2. For the curves given in polar coordinates by $r = 2 \sin \varphi$ and $r = 2 \cos \varphi$:

   (a) prove that both curves are circles and provide a sketch (hint: try multiplying the equation by r and expressing r^2, $r \sin \varphi$ and $r \cos \varphi$ by x and y),

   (b) compute the area of the region that lies inside both circles.

3. Given the differential equation $y' = 2xy^2$ with initial condition $y(0) = 1$

   (a) find its exact solution,

   (b) use Euler's method with step size 0.2 to estimate $y(0.4)$ and compare the result to the exact value $y(0.4)$.

4. Find the general solution of the nonhomogeneous second order linear equation $\ddot{x} + \dot{x} - 2x = t^2$.