Mathematical Modelling Exam

29. 6. 2020

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 105 minutes to solve the problems.

1. **[10 points]** Let A, B be $m \times n$ matrices, $m, n \in \mathbb{N}$, such that $A^T B = 0$ and $BA^T = 0$. Verify the following statements:

   (a) **[1]** Every column of A is perpendicular to every column of B.
   Hint: What is the meaning of the entry in the i-th row and j-th column of $A^T B$?

   (b) **[2]** $A^+ B = B^+ A = 0$.
   Hint: Remember the geometric meaning of $A^+ b$ (resp. $B^+ a$), where b (resp. a) is a column in $\mathbb{R}^m$, and use this for every column of the matrix B (resp. A).

   (c) **[1]** Every column of A^T is perpendicular to every column of B^T.
   Hint: What is the meaning of the entry in the i-th row and j-th column of $(B^T)^T A^T = BA^T$?

   (d) **[2]** $BA^+ = AB^+ = 0$.
   Hint: Assuming (1b) is true, this statement can be proved by plugging A^T and B^T into the appropriate variables in (1b).

   (e) **[4]** $(A + B)^+ = A^+ + B^+$.
   Hint: Use (1b), (1d) in the verification of this part.

   Note: If you are not able to prove (1b), you can assume it is true in proving (1d), and also you can assume both of them are true in proving (1e).

2. **[10 points]** For the parametric curve

$$f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2t - t^2 \\ 3t - t^3 \end{bmatrix}, \quad \text{where} \quad t \in \mathbb{R},$$

   solve the following:

   (a) **[1]** Find intersections with coordinate axes.

   (b) **[1]** Find points at which the tangent is horizontal or vertical.

   (c) **[1]** Find points where $x'(t) = y'(t) = 0$.

   (d) **[1]** Determine the asymptotic behaviour (limits as $t \to \pm\infty$).

   (e) **[2]** Show that there are no self-intersections.

   (f) **[4]** Plot the curve.

   Hint: To notice that the curve does not have any self-intersections verify that $1 - x(t) = 1 - x(s)$ implies $s = 2 - t$ and plug this into the equation $y(t) = y(s)$.

3. **[10 points]** Let

$$F(x, y) := \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} x^2 + y^2 - 10x + y \\ x^2 - y^2 - x + 10y \end{bmatrix}$$

   be a vector function and $a = (2, 4) \in \mathbb{R}^2$ a point.

   (a) **[2]** Calculate the Jacobian matrix of the function F in the point a.

(b) **[3]** Calculate the linear approximation of F in the point a.

(c) **[5]** Perform one step of Newton's method to find the approximate solution of the system

$$F(x, y) = \begin{bmatrix} 1 \\ 25 \end{bmatrix}$$

with the initial approximation a.

4. **[10 points]** Find the solution $[x(t), y(t)]$ of the nonautonomous system of first order linear equations

$$\begin{aligned} \dot{x} &= 2x - y, \\ \dot{y} &= -2x + y + 18t, \end{aligned}$$

which satisfies $x(0) = 1$, $y(0) = 0$.

Hint: One of the particular solutions of the system above is of the form $x(t) = At^2 + Bt + C$, $y(t) = Dt^2 + Et + F$, where A, B, C, D, E, F are constants.