Mathematical Modelling Exam

02. 06. 2021

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Compute the singular value decomposition (SVD) of the matrix

$$B = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

Solution. We have to compute the orthogonal matrices

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

and a diagonal rectangular matrix $\Sigma \in \mathbb{R}^{2\times 3}$ such that

$$B = U\Sigma V^T.$$

We have that

$$BB^T = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix},$$

which implies

$$\det(BB^T - \lambda I_2) = (11-\lambda)^2 - 1 = (11-\lambda-1)(11-\lambda+1) = (12-\lambda)(10-\lambda).$$

So the eigenvalues of BB^T are $\lambda_1 = 12$, $\lambda_2 = 10$ and hence

$$\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}.$$

The kernel of

$$BB^T - \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and hence $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

The kernel of

$$BB^T - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

contains the vector $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and hence $u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

Now the first two columns of V are

$$v_1 = \frac{1}{\sqrt{12}} B^T u_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v_2 = \frac{1}{\sqrt{10}} B^T u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

Finally,

$$v_3 = \frac{v_1 \times v_2}{\|v_1 \times v_2\|} = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$$

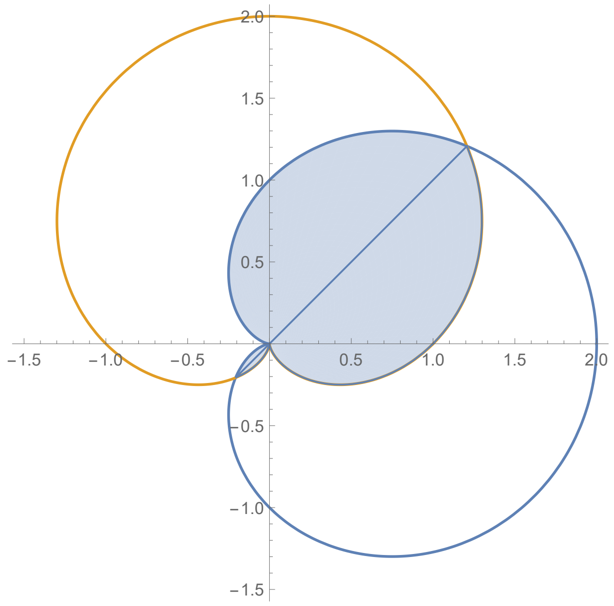
2. Sketch the closed curves given in polar coordinates by

$$r_1(\varphi) = 1 + \cos\varphi \qquad \text{and} \qquad r_2(\varphi) = 1 + \sin\varphi.$$

Compute the area of the intersection of the bounded regions determined by the curves.

Hint: You will need the formulas $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to compute the area.

Solution.



Since $\sin\varphi$ and $\cos\varphi$ are periodic function with period 2π, it is enough to restrict ourselves to the interval $[0, 2\varphi]$. First we have to determine the points, where the curves intersect. As seen from the sketch one of the points is the origin $(0,0)$, where both polar radia are 0. This is true for $\varphi = \pi$ for r_1 and $\varphi = \frac{3\pi}{2}$ for r_2. The other intersections can occur for nonzero radia, in which case they have to be the same for at the same angle. Now:

$$r_1(\varphi) = r_2(\varphi) \quad \Leftrightarrow \quad \sin\varphi = \cos\varphi \quad \Leftrightarrow \quad \varphi \in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}.$$

So the other two intersections are points

$$A = \left(r_1\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right), r_1\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)\right)$$

and

$$B = \left(r_1\left(\frac{5\pi}{4}\right)\cos\left(\frac{5\pi}{4}\right), r_1\left(\frac{5\pi}{4}\right)\sin\left(\frac{5\pi}{4}\right)\right).$$

We see from the sketch that the intersection consists of the area enclosed by r_1 on the interval $[\frac{\pi}{4}, \frac{5\pi}{4}]$ and the area enclosed by r_2 on the union of intervals $[0, \frac{\pi}{4}] \cup [\frac{5\pi}{4}, 2\pi]$. Hence,

$$\text{area} = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (r_1(\varphi))^2\, d\varphi + \frac{1}{2}\int_{\frac{5\pi}{4}}^{2\pi} (r_2(\varphi))^2\, d\varphi + \frac{1}{2}\int_0^{\frac{\pi}{4}} (r_2(\varphi))^2\, d\varphi.$$

Further on,

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1+\cos\varphi)^2\, d\varphi = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1+2\cos\varphi+\cos^2\varphi)\, d\varphi$$

$$= \left[\frac{3}{2}\varphi + 2\sin\varphi + \frac{1}{4}\sin 2\varphi\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{3}{2}\pi - 2\sqrt{2},$$

where we used $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ in the second equality. Similarly

$$\int_{\frac{5\pi}{4}}^{2\pi} (1+\sin\varphi)^2\, d\varphi + \int_0^{\frac{\pi}{4}} (1+\sin\varphi)^2\, d\varphi =$$

$$= \int_{-3\frac{\pi}{4}}^{0} (1+\sin\varphi)^2\, d\varphi + \int_0^{\frac{\pi}{4}} (1+\sin\varphi)^2\, d\varphi$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (1+\sin\varphi)^2\, d\varphi = \left[\frac{3}{2}\varphi - 2\cos\varphi - \frac{1}{4}\sin 2\varphi\right]_{\frac{-3\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{3}{2}\pi - 2\sqrt{2},$$

where we used $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ in the third equality. So, area $= \frac{3}{2}\pi - 2\sqrt{2}$.

3. Solve the differential equation

$$3y'\cos x + y\sin x - \frac{1}{y^2} = 0, \tag{1}$$

given the initial condition $y(0) = 1$.

Hint: Note that this DE can be transformed into a first order linear nonhomogeneous DE by multiplying it with an appropriate factor. To compute $\int \tan x\, dx$ use the substitution $u = \cos x$. Also remember that $\int \frac{1}{(\cos x)^2}\, dx = \tan x + C$.

Solution. Multiplying the DE (1) with y^2 we get

$$3y^2y'\cos x + y^3\sin x - 1 = 0. \tag{2}$$

The homogeneous part $3y^2y'\cos x + y^3\sin x = 0$ can be solved by separation of variables. We get

$$3\frac{dy}{y} = -\tan x\, dx$$

and hence

$$3\log|y| = -\int \tan x\, dx = \int \frac{du}{u} = \log|u| + \log K = \log(K\cos x),$$

where we used the substitution $u = \cos x$ in the third equality and K is a constant. So the solution of the homogeneous part of (2) is

$$y_h(x) = K(\cos x)^{\frac{1}{3}}.$$

To find one particular solution we use variation of constants, i.e.,

$$y_p(x) = K(x)(\cos x)^{\frac{1}{3}}.$$

Plugging $y_p(x)$ into (2) we get

$$3(K(x))^2 K'(x)(\cos x)^2 = 1. \tag{3}$$

(3) is a separable DE:

$$3K^2\, dK = \frac{1}{(\cos x)^2}\, dx \qquad \Rightarrow \qquad K^3 = \tan x.$$

Hence,

$$y_p(x) = (\tan x)^{\frac{1}{3}} \cdot (\cos x)^{\frac{1}{3}} = (\sin x)^{\frac{1}{3}},$$

and the general solution of (2) is

$$y(x) = K(\cos x)^{\frac{1}{3}} + (\sin x)^{\frac{1}{3}}.$$

Using that $y(0) = 1$ we get $K = 1$.

4. Find the general solution of the system

$$\begin{aligned} \dot{x} &= 2x - 3y, \\ \dot{y} &= x - 2y, \end{aligned} \tag{4}$$

and sketch the phase potrait.

Solution. The matrix form of the system (4) is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The eigenvalues of the matrix A are the roots of the determinant:

$$\det\left(A - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 4 = (\lambda - 1)(\lambda + 1).$$

So, $\lambda_1 = 1$ and $\lambda_2 = -1$.

The kernel of

$$A - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$$

contains the vector $u_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$.

The kernel of

$$A - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$$

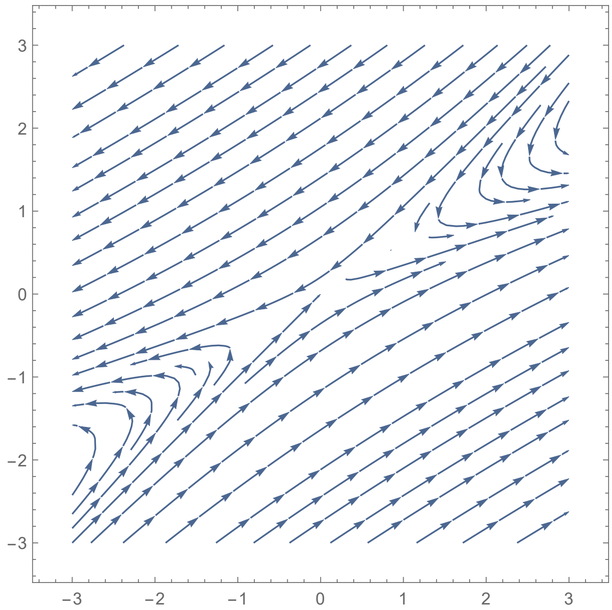
contains the vector $u_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

So, the general solution to (4) is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^t u_1 + C_2 e^{-t} u_2,$$

where C_1 and C_2 are constants.

The phase portait is the following: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