Mathematical Modelling Exam

02. 06. 2021

This is an open book exam. You are allowed to use your notes, books and any other literature. You are NOT allowed to use any communication device. You have 100 minutes to solve the problems.

1. Compute the singular value decomposition of the matrix

$$B = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}.$$

2. Sketch the closed curves given in polar coordinates by

$$r(\varphi) = 1 + \cos\varphi \qquad \text{and} \qquad r(\varphi) = 1 + \sin\varphi.$$

   Compute the area of the intersection of the bounded regions determined by the curves.

   Hint: You will need the formulas $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to compute the area.

3. Solve the differential equation

$$3y' \cos x + y \sin x - \frac{1}{y^2} = 0, \qquad y(0) = 1.$$

   Hint: Note that this DE can be transformed into a first order linear nonhomogeneous DE by multiplying it with an appropriate factor. To compute $\int \tan x \, dx$ use the substitution $u = \cos x$. Also remember that $\int \frac{1}{\cos^2 x} \, dx = \tan x + C$.

4. Find the general solution of the system

$$\begin{aligned} \dot{x} &= 2x - 3y, \\ \dot{y} &= x - 2y, \end{aligned}$$

   and sketch the phase potrait.