Name and surname: ______________________ Student ID: ____________________

	Problem 1	Problem 2	Sum
Points			

Midterm EXAM in Mathematical Modelling
March 12, 2025

For each of tasks 1.-2. show your work and justify all your answers.

1. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be two vectors of norm 1, i.e. $\|u\| = \|v\| = 1$, and let $A = \vec{u}\vec{v}^\mathsf{T} \in \mathbb{R}^{n\times n}$ be a rank 1 matrix. Prove that $B = \vec{v}\vec{u}^\mathsf{T}$ is the Moore-Penrose inverse of A.

Option 1:

$\underbrace{A}_{n\times n} = \vec{u}\vec{v}^T = \underbrace{\vec{u}}_{n\times 1} \cdot \underbrace{[1]}_{\Sigma,\ 1\times 1} \underbrace{\vec{v}^T}_{1\times n}$ is SVD of A

By Theorem $A^+ = \vec{v}\,[1]^+ \vec{u}^T = \vec{v}\,[1]\,\vec{u}^T = \vec{v}\vec{u}^T = B$

Option 2:

(1) $ABA = \vec{u}\underbrace{\vec{v}^T\vec{v}}_{=1}\underbrace{\vec{u}^T\vec{u}}_{=1}\vec{v}^T = \vec{u}\vec{v}^T = A$ ✓

(2) $BAB = \vec{v}\underbrace{\vec{u}^T\vec{u}}_{=1}\underbrace{\vec{v}^T\vec{v}}_{=1}\vec{u}^T = \vec{v}\vec{u}^T = B$ ✓

(3) $(AB) = (\vec{u}\underbrace{\vec{v}^T\vec{v}}_{=1}\vec{u}^T) = \vec{u}\vec{u}^T$ is symmetric

(4) $BA = \vec{v}\vec{u}^T\vec{u}\vec{v}^T = \vec{v}\vec{v}^T$ is symmetric

Option 3: Write A by components and verify (1)-(4) above ☹.

2. Find a point on intersection of planes $x - y = 1$ and $x + z = 2$, that is the closest to the origin $(0,0,0)$.

Intersection of planes is the solution of linear system

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

which is an underdetermined system with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{matrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{matrix}$$

The point closest to the origin is $A^+\vec{b}$.

A is full row rank $\Rightarrow A^+ = A^T(AA^T)^{-1}$

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (AA^T)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} =$$

$$\Rightarrow A^+b = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ point $P(1,0,1)$

2. Find a point on intersection of planes $y - z = 1$ and $x + y = 2$, that is the closest to the origin $(0,0,0)$.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (AA^T)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^+ = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$A^+ b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$P(1,1,0)$

2. Find a point on intersection of planes $x - z = 1$ and $x + y = 2$, that is the closest to the origin $(0,0,0)$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (AA^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^+ = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$A^+ b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$P(1,1,0)$