Mathematical Modelling Exam

June 5th, 2024

You have 90 minutes to solve the problems. The numbers in $[\cdot]$ represent points.

1. Answer the following questions. In YES/NO questions **verify your reasoning**.

   (a) **[2]** Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ a vector. The orthogonal projection of b to the linear span of the columns of A is equal to AA^+b. YES/NO

   Solution: Yes. $x = A^+b$ is a least squares error solution of the system $Ax = b$. This can be equivalently expressed by saying that $A(A^+b)$ is an orthogonal projection of b to $\mathcal{C}(A)$.

   (b) **[2]** Assume that the Gauss–Newton method used to solve an overdetermined system $f(x) = 0$ converges to $\tilde{x} \in \mathbb{R}^n$ for some initial approximation $x^{(0)}$. Then $\tilde{x}$ is a least squares error solution to the system. YES/NO

   Solution: No. Gauss–Newton method can converge to one of the local minima of the function $f_1^2 + \ldots + f_m^2$, where $f_1, \ldots, f_m$ are coordinate functions of f.

   (c) **[2]** Let $\mathcal{C} = \{(x(t), y(t)) : t \in \mathbb{R}\}$ be some curve in the xy–plane. Let $\mathcal{S}$ be a surface obtained by revolving $\mathcal{C}$ around x–axis for 360 degrees. Write down the parametrization of $\mathcal{S}$.

   Solution: $\mathcal{S} = \{(x(t), y(t)\cos v, y(t)\sin v) : t \in \mathbb{R}, v \in [0, 2\pi]\}$.

   (d) **[2]** For every choice of the constants $c_i \in [0,1]$, $a_{ij} \in [0,\infty)$ and $b_i \in [0,1]$, the Runge–Kutta method with a Butcher tableau

$$\begin{array}{c|cccc} 0 & 0 & & & \\ c_2 & a_{21} & 0 & & \\ c_3 & a_{31} & a_{32} & 0 & \\ c_4 & a_{41} & a_{42} & a_{43} & 0 \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array}$$

   for solving the differential equation $y'(x) = f(x,y)$, $y(x_0) = y_0$ will be of order 4. YES/NO

   Solution: No. When deriving the coefficients of the Butcher tableau it is important to make coefficients at h^k, for $k = 1,2,3,4$, in the Taylor expansions of $y(x_n + h)$ and

$$y(x_n) + b_1 \underbrace{hf(x_n, y_n)}_{k_1} + b_2 \underbrace{hf(x_n + c_2 h, y_n + a_{21}k_1)}_{k_2} + b_3 \underbrace{hf(x_n + c_3 h, y_n + a_{31}k_1 + a_{32}k_2)}_{k_3} + b_4 hf(x_n + c_4 h, y_n + a_{41}k_1 + a_{42}k_2 + a_{43}k_3),$$

   equal.

(e) **[2]** Let

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= (*),\\ \dot{x}_3 &= (*),\\ \dot{x}_4 &= -3x_1 + x_2 + 4x_3 + 5x_4,\end{aligned}$$

by a system of differential equations, which comes in a standard way from some higher order differential equation with one dependent variable. What is $(*)$ and what was the original differential equation?

Solution: $\dot{x}_2 = x_3$, $\dot{x}_3 = x_4$. The original DE was $x^{(4)} = -3x + x' + 4x'' + 5x^{(3)}$.

2. We are given points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$. We would like to determine α and β, such that the function

$$f(x) = \cos\alpha \cdot e^{\beta x}$$

fits best to the points in the sense of least squares error.

(a) **[3]** Write down explicitly the nonlinear system we have to solve. Identify the variables.

Solution: We have m conditions $f(x_i) = \cos\alpha \cdot e^{\beta x_i} = y_i$, $i = 1, \ldots, m$. So we are solving the nonlinear system

$$G(\alpha,\beta) = \begin{pmatrix} G_1(\alpha,\beta) \\ G_2(\alpha,\beta) \\ \vdots \\ G_m(\alpha,\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha \cdot e^{\beta x_1} \\ \cos\alpha \cdot e^{\beta x_2} \\ \vdots \\ \cos\alpha \cdot e^{\beta x_m} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

The variables are α, β.

(b) **[4]** Write one step of Gauss–Newton method for solving the problem. Determine the Jacobian matrix needed explicitly.

Solution: For Gauss–Newton method we need initial approximations α_0, β_0 and then each step is equal to

$$\begin{pmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} - (JG(\alpha_n,\beta_n))^{+} G(\alpha_n,\beta_n),$$

where

$$JG(\alpha_n,\beta_n)) = \begin{pmatrix} \frac{\partial G_1(\alpha_n,\beta_n)}{\partial\alpha} & \frac{\partial G_1(\alpha_n,\beta_n)}{\partial\beta} \\ \frac{\partial G_2(\alpha_n,\beta_n)}{\partial\alpha} & \frac{\partial G_2(\alpha_n,\beta_n)}{\partial\beta} \\ \vdots & \\ \frac{\partial G_m(\alpha_n,\beta_n)}{\partial\alpha} & \frac{\partial G_m(\alpha_n,\beta_n)}{\partial\beta} \end{pmatrix} = \begin{pmatrix} -\sin\alpha_n \cdot e^{\beta_n x_1} & \cos\alpha_n \cdot x_1 e^{\beta_n x_1} \\ -\sin\alpha_n \cdot e^{\beta_n x_2} & \cos\alpha_n \cdot x_2 e^{\beta_n x_2} \\ \vdots & \\ -\sin\alpha_n \cdot e^{\beta_n x_m} & \cos\alpha_n \cdot x_m e^{\beta_n x_m} \end{pmatrix}$$

(c) **[3]** Compute $J^T J$ explicitly, where J is the Jacobian from the previous question and comment on the efficient way of computing J^+. You do not need to compute J^+ explicitly.

Solution: We have that

$$
\begin{aligned}
&(JG(\alpha_n,\beta_n)))^T JG(\alpha_n,\beta_n)) \\
&= \begin{pmatrix} \sin^2\alpha_n \cdot \sum_{i=1}^m e^{2\beta_n x_i} & -\sin\alpha_n\cos\alpha_n \cdot \sum_{i=1}^m x_i e^{2\beta_n x_i} \\ -\sin\alpha_n\cos\alpha_n \cdot \sum_{i=1}^m x_i e^{2\beta_n x_i} & \cos^2\alpha_n \cdot \sum_{i=1}^m x_i^2 e^{2\beta_n x_i} \end{pmatrix}.
\end{aligned}
$$

So efficient way of computing $(JG(\alpha_n,\beta_n))^+$ is to compute

$$\left((JG(\alpha_n,\beta_n)))^T JG(\alpha_n,\beta_n)\right)^{-1} JG(\alpha_n,\beta_n)^T.$$

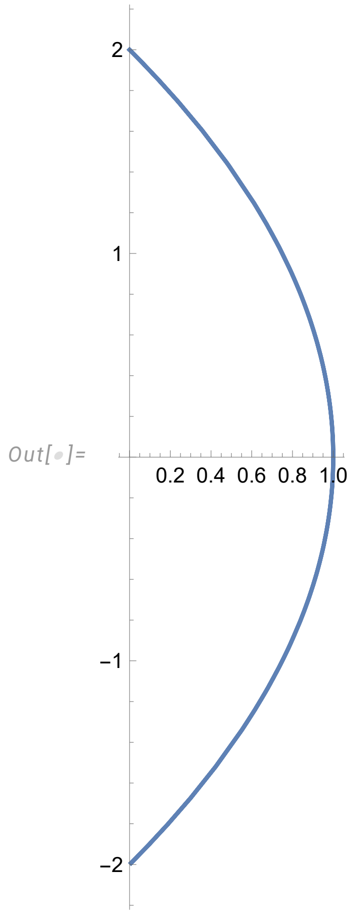
3. Let

$$x(t) = \sin^2 t, \quad y(t) = 2\cos t, \quad t \in \mathbb{R}$$

be a curve.

(a) **[7]** Sketch the curve. Determine also all local extrema in x and y direction and all intersections with the axes.

Solution:



We have:

i. $x(t) = 0 \Leftrightarrow \sin^2 t = 0 \Leftrightarrow t = k\pi, k \in \mathbb{Z}$. So intersections with the y axis are $(0, 2)$ (for $k \in 2\mathbb{Z}$) and $(0, -2)$ (for $k \in 2\mathbb{Z} + 1$).

ii. $y(t) = 0 \Leftrightarrow 2\cos t = 0 \Leftrightarrow t = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$. So the intersection with the x axis is $(0, 1)$.

iii. $x'(t) = 0 \Leftrightarrow 2\sin t \cos t = \sin 2t = 0 \Leftrightarrow t \in \{k\pi, \frac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$. So local maxima of x are equal to 1 for $t = k\pi$, $k \in \mathbb{Z}$, while local minima of x are 0 for $t = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

iv. $y'(t) = 0 \Leftrightarrow -2\sin t = 0 \Leftrightarrow t = k\pi, k \in \mathbb{Z}$. So local maxima of y are equal to 2 for $t = 2k\pi$, $k \in \mathbb{Z}$, while local minima of y are -2 for $t = \pi + 2k\pi$, $k \in \mathbb{Z}$.

(b) **[5]** Compute the length of the trace of the curve.

Solution: Notice that the trace of the whole trace of the curve is obtained already when restricting t to $[0, \pi]$. So

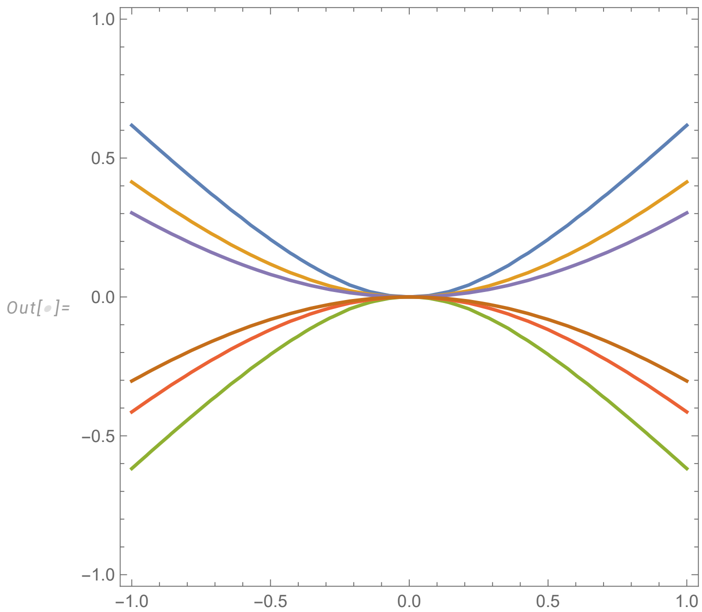
$$\ell = \int_0^\pi \|f'(t)\| dt = \int_0^\pi \left\| \begin{pmatrix} \sin 2t \\ -2\sin t \end{pmatrix} \right\| dt = \int_0^\pi \sqrt{\sin^2(2t) + 4\sin^2 t} dt$$
$$= \int_0^\pi \sqrt{4\sin^2 t(\cos^2 t + 1)} dt = \int_0^\pi 2\sin t\sqrt{\cos^2 t + 1} dt = 4\int_0^{\pi/2} \sin t\sqrt{\cos^2 t + 1} dt$$
$$= 4\int_0^1 \sqrt{u^2 + 1} du = 4\Big[\frac{1}{2}\log(u + \sqrt{1 + u^2}) + \frac{1}{2}u\sqrt{1 + u^2}\Big]_0^1 = 2\log(1 + \sqrt{2}) + 2\sqrt{2},$$

where we introduced a substitution $u = \cos t$.

4. Let $x^2 - y^2 = ay$, $a \in \mathbb{R}$, be a family of curves.

(a) **[5]** Plot a few members of the family.

Solution:



(b) **[5]** Derive a differential equation determining orthogonal trajectories to the given family.

Solution: By differentiang w.r.t. x we have $2x - 2yy' = ay'$. Hence, $a = \frac{2x}{y'} - 2y$. Using this in the original equation we get $x^2 - y^2 = \frac{2xy}{y'} - 2y^2$. Since for orthogonal trajectories y' is $-\frac{1}{y'}$, we get the DE $-2xyy' = x^2 + y^2$.

(c) **[5]** Solve the differential equation obtained.

Solution: We notice that the DE is homogeneous, i.e., $y' = -\frac{1}{2}\frac{x}{y} - \frac{1}{2}\frac{y}{x}$. To solve the DE obtained we use a new variable $u = \frac{y}{x}$. Hence, $xu = y$ and $u + xu' = y'$. So the DE becomes $u + xu' = -\frac{1}{2u} - \frac{1}{2}u$. Hence, $\frac{2u}{3u^2+1}du = -\frac{1}{x}dx$. Introducing a new variable $v = 3u^2 + 1$ we have $\frac{1}{3v}dv = -\frac{1}{x}dx$. Hence, $\frac{1}{3}\log|v| = -\log K|x|$ and $v = \frac{K}{x^3}$. It follows that $3u^2 + 1 = 3\frac{y^2}{x^2} + 1 = \frac{K}{x^3}$. Finally, $3y^2x + x^3 = K$, $K \in \mathbb{R}$.