Name and surname: ____________________ Student ID: ____________________

	Problem 1	Problem 2	Sum
Points	5,5	7	12,5

Midterm EXAM in Mathematical Modelling
March 12, 2025

For each of tasks 1.-2. show your work and justify all your answers.

1. Let $\vec{F}\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a vector function defined by

$$\vec{F}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} \\ \sqrt[3]{xyz} \end{bmatrix}.$$

Using linear approximation at $(1,1,-1)$, compute the approximate value of $\vec{F}(1.1, 0.9, -0.9)$.

$$J_{\vec{F}}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} & 2e^{x+2y+3z} & 3e^{x+2y+3z} \\ \frac{yz}{3\sqrt[3]{(xyz)^2}} & \frac{xz}{3\sqrt[3]{(xyz)^2}} & \frac{xy}{3\sqrt[3]{(xyz)^2}} \end{bmatrix}$$

2

1

$a = (1,1,-1)$

1

$$\vec{F}(a) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad J_{\vec{F}}(a) = \begin{bmatrix} 1 & 2 & 3 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\vec{h} = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.1 \end{bmatrix}$$

0,5

$$\vec{F}(\vec{a}+\vec{h}) \approx \vec{F}(\vec{a}) + J_F(\vec{a})\cdot\vec{h} =$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0,1 \\ -0,1 \\ 0,1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0,2 \\ \frac{1}{30} \end{bmatrix} = \begin{bmatrix} 1,2 \\ -\frac{29}{30} \end{bmatrix}$$

1

Name and surname: ______________________ Student ID: ____________________

	Problem 1	Problem 2	Sum
Points			

Midterm EXAM in Mathematical Modelling
March 12, 2025

For each of tasks 1.-2. show your work and justify all your answers.

1. Let $\vec{F}\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a vector function defined by

$$\vec{F}(x,y,z) = \begin{bmatrix} e^{3x+2y+z} \\ \sqrt[3]{xyz} \end{bmatrix}.$$

-1 $-0{,}9$

Using linear approximation at $(1, 1, -1)$, compute the approximate value of $\vec{F}(1.1, \cancel{1.1}, -0.9)$.

$$J_{\vec F}(x,y,z) = \begin{bmatrix} 3e^{3x+2y+z} & 2e^{3x+2y+z} & e^{3x+2y+z} \\ \frac{yz}{3\sqrt[3]{(xyz)^2}} & \frac{xz}{3\sqrt[3]{(xyz)^2}} & \frac{xy}{3\sqrt[3]{(xyz)^2}} \end{bmatrix}$$

$\underline{a} = (1, -1, -1)$

$$\vec F(\underline{a}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad J_{\vec F}(\underline{a}) = \begin{bmatrix} 3 & 2 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\vec h = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$

$$\vec F(\vec a + \vec h) \approx \vec F(\vec a) + J_{\vec F}(\vec a) \cdot \vec h =$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0{,}1 \\ 0{,}1 \\ 0{,}1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0{,}6 \\ -\frac{1}{30} \end{bmatrix} = \begin{bmatrix} 1{,}6 \\ \frac{29}{30} \end{bmatrix}$$

Name and surname: ______________________ Student ID: __________________

	Problem 1	Problem 2	Sum
Points			

Midterm EXAM in Mathematical Modelling
March 26, 2025

For each of tasks 1.-2. show your work and justify all your answers.

1. Let $\vec{F}\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a vector function defined by

$$\vec{F}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} \\ \sqrt[3]{xyz} \end{bmatrix}.$$

Using linear approximation at $(1,1,-1)$, compute the approximate value of $\vec{F}(1.1, 1.1, -0.9)$.

$$J_{\vec{F}}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} & 2e^{x+2y+3z} & 3e^{x+2y+3z} \\ \frac{yz}{3\sqrt[3]{(xyz)^2}} & \frac{xz}{3\sqrt[3]{(xyz)^2}} & \frac{xy}{3\sqrt[3]{(xyz)^2}} \end{bmatrix}$$

$\underline{a} = (1,1,-1)$

$$\vec{F}(\underline{a}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad J_{\vec{F}}(\underline{a}) = \begin{bmatrix} 1 & 2 & 3 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\vec{h} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$

$$\vec{F}(\vec{a}+\vec{h}) \approx \vec{F}(\vec{a}) + J_F(\vec{a})\cdot\vec{h} =$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0,1 \\ 0,1 \\ 0,1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0,6 \\ -\frac{1}{30} \end{bmatrix} = \begin{bmatrix} 1,6 \\ -\frac{31}{30} \end{bmatrix}$$

2. Let $\vec{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a vector function defined by

$$\vec{G}(x,y,z) = \begin{bmatrix} e^{3x+2y+z} \\ \sqrt[3]{xyz} \\ x \end{bmatrix}.$$

Using one step of Newton's method, approximate the solution of the equation $\vec{G}(x,y,z) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ with the initial approximation $(x_0, y_0, z_0) = (1, \overset{-1}{\cancel{1}}, -1)$.

$$\vec{H}(x,y,z) = \begin{bmatrix} e^{3x+2y+z} - 2 \\ \sqrt[3]{xyz} + 2 \\ x - 1 \end{bmatrix}$$ Looking for $\vec{H}(x,y,z) = \vec{0}$.

$$J_{\vec{H}}(x,y,z) = \begin{bmatrix} 3e^{3x+2y+z} & 2e^{3x+2y+z} & e^{3x+2y+z} \\ \frac{yz}{3\sqrt[3]{(xyz)^2}} & \frac{xz}{3\sqrt[3]{(xyz)^2}} & \frac{xy}{3\sqrt[3]{(xyz)^2}} \\ 1 & 0 & 0 \end{bmatrix},$$

$$J_{\vec{H}}(1,-1,-1) = \begin{bmatrix} 3 & 2 & 1 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix} \qquad J_{\vec{H}}(1,-1,-1)^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & -4 \\ -1 & -6 & 5 \end{bmatrix}$$

$$\vec{H}(1,-1,-1) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - J_{\vec{H}}(1,1,-1)^{-1} \cdot \vec{H}(1,1,-1) =$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & -4 \\ -1 & -6 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 8 \\ -17 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 16 \end{bmatrix}$$

2. Let $\vec{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a vector function defined by

$$\vec{G}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} \\ \sqrt[3]{xyz} \\ z \end{bmatrix}.$$

Using one step of Newton's method, approximate the solution of the equation $\vec{G}(x,y,z) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ with the initial approximation $(x_0, y_0, z_0) = (1, 1, -1)$.

1 $\to$ $$\vec{H}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} - 2 \\ \sqrt[3]{xyz} + 2 \\ z - 1 \end{bmatrix}$$ Looking for $\vec{H}(x,y,z) = \vec{0}$.

2 $\to$ $$J_{\vec{H}}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} & 2e^{x+2y+3z} & 3e^{x+2y+3z} \\ \frac{yz}{3\sqrt[3]{(xyz)^2}} & \frac{xz}{3\sqrt[3]{(xyz)^2}} & \frac{xy}{3\sqrt[3]{(xyz)^2}} \\ 0 & 0 & 1 \end{bmatrix},$$

$$J_{\vec{H}}(1,1,-1) = \begin{bmatrix} 1 & 2 & 3 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ -1 & -1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 1 & 1 & 3 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to$$

$$\vec{H}(1,1,-1) = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

2 $$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -6 & 5 \\ 0 & 1 & 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \begin{bmatrix} -1 & -6 & 5 \\ 1 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix} = J_{\vec{H}}(1,1,-1)^{-1}$$

1 $\to$ $$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - J_{\vec{H}}(1,1,-1)^{-1} \cdot \vec{H}(1,1,-1) =$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 & -6 & 5 \\ 1 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -15 \\ 10 \\ -2 \end{bmatrix} = \begin{bmatrix} 16 \\ -9 \\ 1 \end{bmatrix}$$

1

2. Let $\vec{G}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a vector function defined by

$$\vec{G}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} \\ \sqrt[3]{xyz} \\ y \end{bmatrix}.$$

Using one step of Newton's method, approximate the solution of the equation $\vec{G}(x,y,z) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ with the initial approximation $(x_0, y_0, z_0) = (1, 1, -1)$.

$$\vec{H}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} - 2 \\ \sqrt[3]{xyz} + 2 \\ y - 1 \end{bmatrix}$$

Looking for $\vec{H}(x,y,z) = \vec{0}$.

$$J_{\vec H}(x,y,z) = \begin{bmatrix} e^{x+2y+3z} & 2e^{x+2y+3z} & 3e^{x+2y+3z} \\ \dfrac{yz}{3\sqrt[3]{(xyz)^2}} & \dfrac{xz}{3\sqrt[3]{(xyz)^2}} & \dfrac{xy}{3\sqrt[3]{(xyz)^2}} \\ 0 & 1 & 0 \end{bmatrix},$$

$$J_{\vec H}(1,1,-1) = \begin{bmatrix} 1 & 2 & 3 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 3 & 0 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & 3 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\boxed{\begin{matrix} 1/4 & -9/4 & -5/4 \\ 0 & 0 & 1 \\ 1/4 & 3/4 & -1/4 \end{matrix}}}_{J_{\vec H}(1,1,-1)^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$\vec{H}(1,1,-1) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - J_{\vec H}(1,1,-1)^{-1} \cdot \vec{H}(1,1,-1) =$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 & -9 & -5 \\ 0 & 0 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} -10 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 + \frac{5}{2} \\ 1 \\ -1 - \frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{7}{2} \\ 1 \\ -\frac{3}{2} \end{bmatrix}$$