Name and surname: ______________________ Student ID: __________________

	Problem 1	Problem 2	Sum
Points			

Midterm EXAM in Mathematical Modelling
May 7, 2025

For each of tasks justify all your answers.

1. Find the curve in $\mathbb{R}^2$ that goes through $(2, \mathbf{2})$ and has the following property: its tangent vector at each point (x, y) has the first component equal to $x + y$ and the second component equal to $-y$.

curve $\mathcal{C} \subseteq \mathbb{R}^2$, parametrization as $\vec{F}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ with $\vec{F}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (*)

this can be chosen arbitrary

$\vec{F}'(t) = \begin{bmatrix} x+y \\ -y \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} x+y \\ -y \end{bmatrix}$ system of DE, 2 variables

$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ in matrix form

$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

eigenvalues $1, -1$

$\lambda_1 = 1$: $A - I = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$ $A + I = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} C_1 e^{t} + C_2 e^{-t} \\ -2C_2 e^{-t} \end{bmatrix}$

(*) $2 = x(0) = C_1 + C_2$

$2 = y(0) = -2C_2$ $\Rightarrow C_2 = -1 \Rightarrow C_1 = 3$

$\Rightarrow \vec{F}(t) = \begin{bmatrix} 3e^{t} - e^{-t} \\ 2e^{-t} \end{bmatrix}, \quad t \in \mathbb{R}$

2. Find the solution $y = y(x)$ of the differential equation

$$y'' + 4y' + 4y = \cos(x) \quad \} \; (*)$$

with initial condition $y(0) = y'(0) = 0$.

Finding particular solution:

Guess a particular solution:

$y = y(x) = A\cos x + B\sin x$

$y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$(*)$: $-A\cos x - B\sin x - 4A\sin x + 4B\cos x + 4A\cos x + 4B\sin x = \cos x$ for all $x \in \mathbb{R}$

$\cos x(-A + 4B + 4A - 1) + \sin x(-B - 4A + 4B) = 0 \quad \forall x$

$(3A + 4B - 1)\cos x + (3B - 4A)\sin x = 0 \quad \forall x$

$x = 0$: $(3A + 4B - 1) \cdot 1 + (3B - 4A) \cdot 0 = 0$

$\boxed{3A + 4B = 1}$

$x = \frac{\pi}{2}$: $(3A + 4B - 1) \cdot 0 + (3B - 4A) \cdot 1 = 0$

$\boxed{3B - 4A = 0}$

(alternative argument: sinx and cosx are lin. independent functions)

$$\begin{aligned} 3A + 4B &= 1 \quad /\cdot 4 \\ -4A + 3B &= 0 \quad /\cdot 3 \end{aligned} \Big\} +$$

$25B = 4$

$B = \frac{4}{25}, \; A = \frac{3}{25}$

$\Rightarrow y_P(x) = \frac{3}{25}\cos x + \frac{4}{25}\sin x$

Solving homogeneous part:

$y'' + 4y' + y = 0 \quad \rightsquigarrow \quad \lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)^2 = 0$

$\lambda_1 = \lambda_2 = -2$

$y_H(x) = C_1 e^{-2x} + C_2 x e^{-2x}$

$y(x) = y_H(x) + y_P(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{3}{25}\cos x + \frac{4}{25}\sin x$

$y'(x) = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x} - \frac{3}{25}\sin x + \frac{4}{25}\cos x$

$0 = y(0) = C_1 + \frac{3}{25} \Rightarrow C_1 = -\frac{3}{25}$

$0 = y'(0) = -2C_1 + C_2 + \frac{4}{25} \Rightarrow C_2 = -\frac{6}{25} - \frac{4}{25} = -\frac{10}{25}$

$y(x) = -\frac{3}{25}e^{-2x} - \frac{2}{5}x e^{-2x} + \frac{3}{25}\cos x + \frac{4}{25}\sin x$

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	Problem 1	Problem 2	Sum
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Midterm EXAM in Mathematical Modelling
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For each of tasks justify all your answers.

1. Find the curve in $\mathbb{R}^2$ that goes through $(2,3)$ and has the following property: its tangent vector at each point (x,y) has the first component equal to $2x+y$ and the second component equal to $-y$.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\lambda_1 = 2 \quad : \quad \vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1 \quad : \quad \vec{v_2} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} + C_2 e^{-t} \\ -3C_2 e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} C_1 + C_2 \\ -3C_2 \end{bmatrix} \qquad \begin{matrix} C_2 = -1 \\ C_1 = 3 \end{matrix}$$

$$\Rightarrow \quad \vec{F}(t) = \begin{bmatrix} 3e^{2t} - e^{-t} \\ 3e^{-t} \end{bmatrix}$$

2. Find the solution $y = y(x)$ of the differential equation

$$y'' + 4y' + 4y = \sin(x)$$

with initial condition $y(0) = y'(0) = 0$.

Finding particular solution

Guess a particular solution:

$y = y(x) = A\cos x + B\sin x$

$y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

$(*)$: $-A\cos x - B\sin x - 4A\sin x + 4B\cos x + 4A\cos x + 4B\sin x = \sin x$ for all $x \in \mathbb{R}$

$\cos x(-A + 4B + 4A) + \sin x(-1 - B - 4A + 4B) = 0 \quad \forall x$

$(3A+4B)\cos x + (3B-4A-1)\sin x = 0 \quad \forall x$

$x = 0$: $(3A+4B)\cdot 1 + (3B-4A-1)\cdot 0 = 0$

$\boxed{3A+4B=0}$

$x = \frac{\pi}{2}$: $(3A+4B-1)\cdot 0 + (3B-4A-1)\cdot 1 = 0$

$\boxed{3B-4A=1}$

(alternative argument: sinx and cosx are lin. independent functions)

$$\begin{aligned} 3A+4B &= 0 \quad /\cdot 4 \\ -4A+3B &= 1 \quad /\cdot 3 \quad)+ \\ \hline 25B &= 3 \end{aligned}$$

$B = \frac{3}{25}$, $A = -\frac{4}{25}$

$\Rightarrow y_P(x) = -\frac{4}{25}\cos x + \frac{3}{25}\sin x$

Solving homogeneous part:

$y'' + 4y' + y = 0 \rightsquigarrow \lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0$

$\lambda_1 = \lambda_2 = -2$

$y_H(x) = C_1 e^{-2x} + C_2 x e^{-2x}$

$y(x) = y_H(x) + y_P(x) = C_1 e^{-2x} + C_2 x e^{-2x} - \frac{4}{25}\cos x + \frac{3}{25}\sin x$

$y'(x) = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x} + \frac{4}{25}\sin x + \frac{3}{25}\cos x$

$0 = y(0) = C_1 - \frac{4}{25} \Rightarrow C_1 = +\frac{4}{25}$

$0 = y'(0) = -2C_1 + C_2 + \frac{3}{25} \Rightarrow C_2 = +\frac{8}{25} - \frac{3}{25} = \frac{5}{25}$

$y(x) = +\frac{4}{25}e^{-2x} + \frac{1}{5}xe^{-2x} - \frac{4}{25}\cos x + \frac{3}{25}\sin x$

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For each of tasks justify all your answers.

1. Find the curve in $\mathbb{R}^2$ that goes through $(2, 3)$ and has the following property: its tangent vector at each point (x, y) has the first component equal to $-x$ and the second component equal to $x + y$.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\lambda_1 = 1 : \quad \vec{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 : \quad \vec{v_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2C_2 e^{-t} \\ C_1 e^{t} - C_2 e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2C_2 \\ C_1 - C_2 \end{bmatrix} \qquad \begin{matrix} C_2 = 1 \\ C_1 = 4 \end{matrix}$$

$$\Rightarrow \quad \vec{F}(t) = \begin{bmatrix} 2e^{-t} \\ 4e^{t} - e^{-t} \end{bmatrix}$$

2. Find the solution $y = y(x)$ of the differential equation

$$y'' + 4y' + 4y = -\sin(x)$$

with initial condition $y(0) = y'(0) = 0$.

Finding particular solution

Guess a particular solution:

$$y = y(x) = A\cos x + B\sin x$$
$$y' = -A\sin x + B\cos x$$
$$y'' = -A\cos x - B\sin x$$

(*): $-A\cos x - B\sin x - 4A\sin x + 4B\cos x + 4A\cos x + 4B\sin x = -\sin x$ for all $x \in \mathbb{R}$

$$\cos x(-A + 4B + 4A) + \sin x(+1 - B - 4A + 4B) = 0 \quad \forall x$$

$$(3A+4B)\cos x + (3B-4A+1)\sin x = 0 \quad \forall x$$

$x = 0$: $(3A+4B)\cdot 1 + (3B-4A+1)\cdot 0 = 0$

$$\boxed{3A + 4B = 0}$$

$x = \frac{\pi}{2}$: $(3A+4B-1)\cdot 0 + (3B-4A+1)\cdot 1 = 0$

$$\boxed{3B - 4A = -1}$$

(alternative argument: sinx and cos x are lin. independent functions)

$$\begin{aligned} 3A + 4B &= 0 \quad /\cdot 4 \\ -4A + 3B &= -1 \quad /\cdot 3 \end{aligned} \Big) +$$

$$25B = -3$$

$$B = -\frac{3}{25}, \quad A = +\frac{4}{25}$$

$$\Rightarrow y_P(x) = \frac{4}{25}\cos x - \frac{3}{25}\sin x$$

Solving homogeneous part:

$$y'' + 4y' + y = 0 \rightsquigarrow \lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0$$

$$\lambda_1 = \lambda_2 = -2$$

$$y_H(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

$$y(x) = y_H(x) + y_P(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{4}{25}\cos x - \frac{3}{25}\sin x$$

$$y'(x) = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x} - \frac{4}{25}\sin x - \frac{3}{25}\cos x$$

$$0 = y(0) = C_1 + \frac{4}{25} \Rightarrow C_1 = -\frac{4}{25}$$

$$0 = y'(0) = -2C_1 + C_2 - \frac{3}{25} \Rightarrow C_2 = -\frac{8}{25} + \frac{3}{25} = -\frac{5}{25}$$

$$y(x) = -\frac{4}{25}e^{-2x} - \frac{1}{5}xe^{-2x} + \frac{4}{25}\cos x - \frac{3}{25}\sin x$$

Name and surname: ________________________ Student ID: ____________________

	Problem 1	Problem 2	Sum
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Midterm EXAM in Mathematical Modelling
May 7, 2025

For each of tasks justify all your answers.

1. Find the curve in $\mathbb{R}^2$ that goes through (3, 2) and has the following property: its tangent vector at each point (x, y) has the first component equal to $-x$ and the second component equal to $x + 2y$.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\lambda_1 = 2 \; : \; \vec{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \; : \; \vec{v_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3C_2 e^{-t} \\ C_1 e^{2t} - C_2 e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3C_2 \\ C_1 - C_2 \end{bmatrix} \qquad \begin{matrix} C_2 = 1 \\ C_1 = 3 \end{matrix}$$

$$\Rightarrow \quad \vec{F}(t) = \begin{bmatrix} 3e^{-t} \\ 3e^{2t} - e^{-t} \end{bmatrix}$$

2. Find the solution $y = y(x)$ of the differential equation

$$y'' + 4y' + 4y = -\cos(x)$$

with initial condition $y(0) = y'(0) = 0$.

Finding particular solution

Guess a particular solution:

$y = y(x) = A\cos x + B\sin x$

$y' = -A\sin x + B\cos x$

$y'' = -A\cos x - B\sin x$

(*): $-A\cos x - B\sin x - 4A\sin x + 4B\cos x + 4A\cos x + 4B\sin x = -\cos x$ for all $x \in \mathbb{R}$

$\cos x(-A + 4B + 4A + 1) + \sin x(-B - 4A + 4B) = 0 \quad \forall x$

$(3A + 4B + 1)\cos x + (3B - 4A)\sin x = 0 \quad \forall x$

$x = 0$: $(3A + 4B + 1)\cdot 1 + (3B - 4A)\cdot 0 = 0$

$\boxed{3A + 4B = -1}$

$x = \frac{\pi}{2}$: $(3A + 4B - 1)\cdot 0 + (3B - 4A)\cdot 1 = 0$

$\boxed{3B - 4A = 0}$

(alternative argument: sinx and cos x are lin. independent functions)

$$\begin{aligned} 3A + 4B &= -1 \quad / \cdot 4 \\ -4A + 3B &= 0 \quad / \cdot 3 \end{aligned} \Big) +$$

$25B = -4$

$B = -\frac{4}{25}, \; A = -\frac{3}{25}$

$\Rightarrow y_P(x) = -\frac{3}{25}\cos x - \frac{4}{25}\sin x$

Solving homogeneous part:

$y'' + 4y' + y = 0 \rightsquigarrow \lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)^2 = 0$

$\lambda_1 = \lambda_2 = -2$

$y_H(x) = C_1 e^{-2x} + C_2 x e^{-2x}$

$y(x) = y_H(x) + y_P(x) = C_1 e^{-2x} + C_2 x e^{-2x} - \frac{3}{25}\cos x - \frac{4}{25}\sin x$

$y'(x) = -2C_1 e^{-2x} + C_2 e^{-2x} - 2C_2 x e^{-2x} + \frac{3}{25}\sin x - \frac{4}{25}\cos x$

$0 = y(0) = C_1 - \frac{3}{25} \Rightarrow C_1 = \frac{3}{25}$

$0 = y'(0) = -2C_1 + C_2 - \frac{4}{25} \Rightarrow C_2 = \frac{6}{25} + \frac{4}{25} = \frac{10}{25}$

$y(x) = +\frac{3}{25}e^{-2x} + \frac{2}{5}xe^{-2x} - \frac{3}{25}\cos x - \frac{4}{25}\sin x$