Name and surname: ______________________ Student ID: __________________

	Problem a.	Problem b.	Problem c.	Problem d.	Problem e.	Sum
Points						

Midterm EXAM in Mathematical Modelling
April 9, 2025

For each of tasks justify all your answers.

Let the curve $\mathcal{C}$ be parametrized by function $\vec{F}: \mathbb{R} \to \mathbb{R}^2$, $\vec{F}(t) = \begin{bmatrix} t^2 \\ t^3 - 9t \end{bmatrix}$.

a. Prove that $\mathcal{C}$ has a self-intersection.

Is $\vec{F}(t) = \vec{F}(s)$ for $t \neq s$?

$t^2 = s^2 \Rightarrow s = \pm t \Rightarrow s = -t$

$t^3 - 9t = s^3 - 9s$

$t^3 - 9t = -t^3 + 9t$

$2(t^3 - 9t) = 0$

$t(t^2 - 9) = 0$

$t \neq 0$ (since otherwise $s = 0$)

$(t-3)(t+3) = 0$

$t = \pm 3$ and $s = \mp 3$

$P_{t=3}(9,0)$, $P_{t=-3}(9,0)$

$\Rightarrow$ $\mathcal{C}$ has a self-intersection at $t=3$ and $t=-3$ in point $(9,0)$.

b. Prove that the curve is symmetric over x-axis.

$\vec{F}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, where $x(t) = t^2$ is an even function $(x(-t) = (-t)^2 = t^2 = x(t))$

and $y(t) = t^3 - 9t$ is an odd function, $(y(-t) = (-t)^3 - 9(-t) = -(t^3 - 9t) = -y(t))$

and so $\vec{F}(-t) = \begin{bmatrix} x(-t) \\ y(-t) \end{bmatrix} = \begin{bmatrix} x(t) \\ -y(t) \end{bmatrix}$, which is

$\vec{F}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, reflected over x-axis

c. Compute the following and write the results in the table:

1 Intersections of $\mathcal{C}$ with x-axis	$\overset{t=-3 \,\&\, t=3}{(9,0)}$, $(0,0)$
2 Intersections of $\mathcal{C}$ with y-axis	$(0,0)$
3 Points on $\mathcal{C}$ with horizontal tangents	$(3,6\sqrt{3})$, $(3,-6\sqrt{3})$
4 Points on $\mathcal{C}$ with vertical tangents	$(0,0)$
5 Slopes of tangents on $\mathcal{C}$ in self-intersections	3, -3

1 $y(t)=0 \iff t^3-9t=0 \iff t(t-3)(t+3)=0$

$t=0,\ t=3,\ t=-3$

$P_{-3}(9,0),\ P_0(0,0),\ P_3(9,0)$

2 $x(t)=0 \iff t^2=0 \iff t=0 \qquad P_0(0,0)$

3 $y'(t)=0 \iff x=3,\ t=\pm\sqrt{3}$ (see part b.)

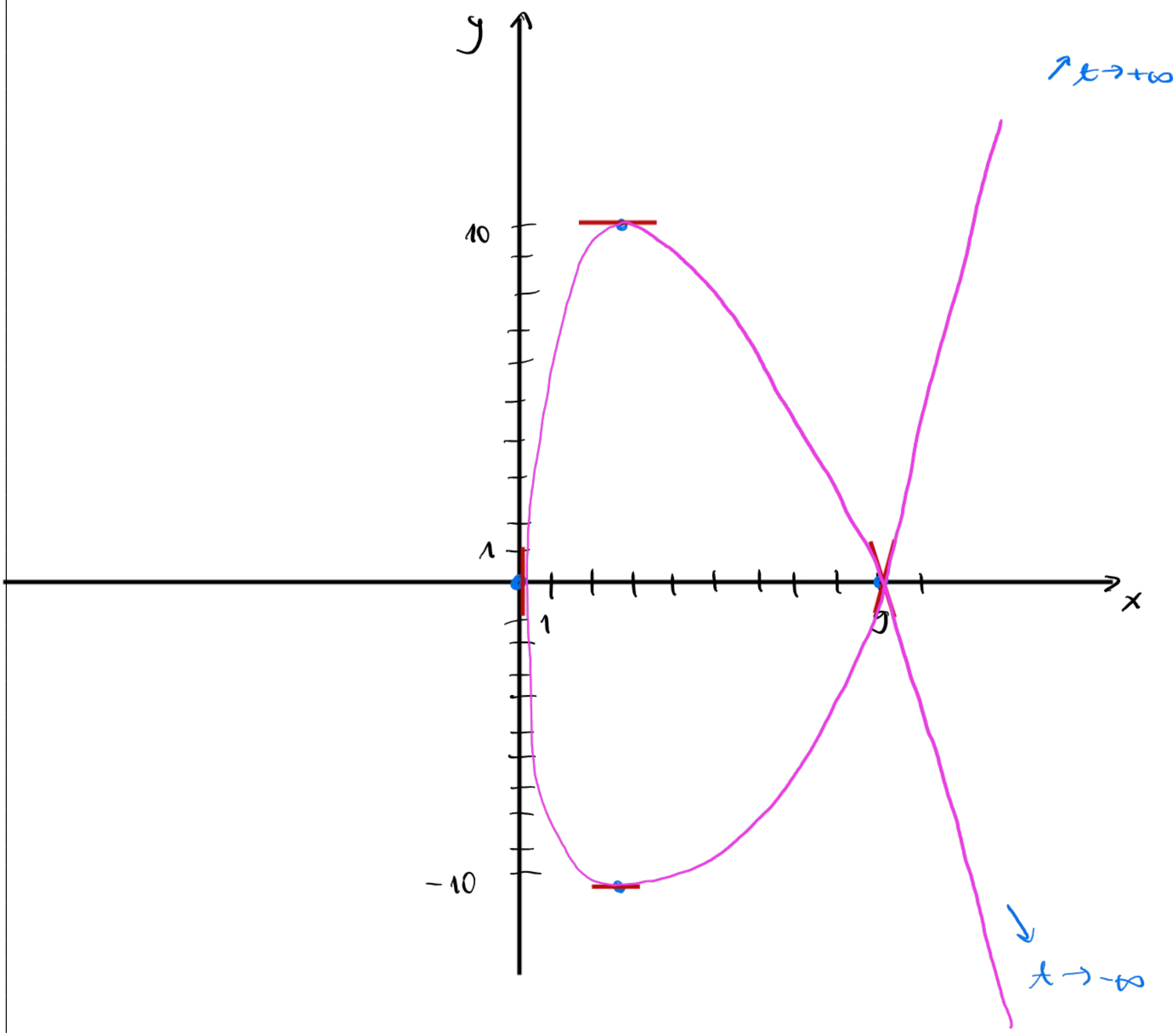
$P_{-\sqrt{3}}(3,\underbrace{6\sqrt{3}}_{\approx 10}),\ P_{\sqrt{3}}(3,-6\sqrt{3})$

4 $x'(t)=0 \iff 2t=0 \iff t=0 \qquad P_0(0,0)$

5 $\dfrac{y'(t)}{x'(t)} = \dfrac{3t^2-9}{2t} \xrightarrow[\text{(see part a.)}]{t\to 3} \dfrac{27-9}{6}=\dfrac{18}{6}=3$

$\xrightarrow{t=-3} \dfrac{27-9}{-6}=-3$

d. Sketch the curve $\mathcal{C}$.



$$x(t) \xrightarrow{t\to\infty} +\infty$$
$$x(t) \xrightarrow{t\to-\infty} +\infty$$

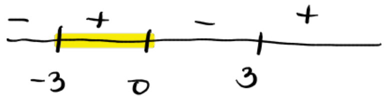
$$y(t) \xrightarrow{t\to\infty} +\infty$$
$$y(t) \xrightarrow{t\to-\infty} -\infty$$

e. Compute the area of the loop of $\mathcal{C}$.

$\vec{F}(t) = \begin{bmatrix} t^2 \\ t^3 - 9t \end{bmatrix}$ has self-intersection at $t = 3$ and $t = -3$ (see part a.)

Since $\vec{F}(-t)$ is reflected $\vec{F}(t)$ over x-axis (see part b.)

$y(t) = t(t-3)(t+3)$



$$\text{area} = 2\int_{-3}^{0} |\underbrace{t^3 - 9t}_{\geq 0}|\, 2t\, dt = \quad \text{for } t \in (-3, 0)$$

$$= 4\int_{-3}^{0} (t^3 - 9t)\, t\, dt$$

$$= 4\int_{-3}^{0} (t^4 - 9t^2)\, dt$$

$$= 4\left(\frac{t^5}{5} - 9\frac{t^3}{3}\right)\Bigg|_{-3}^{0} =$$

$$= \frac{4}{5} t^3 (t^2 - 15) = 0 - \frac{4}{5}(-27)(9 - 15) =$$

$$= -\frac{4 \cdot 27 \cdot 6}{5} = -\frac{2^3 \cdot 3^4}{5}$$

this one is because $x'(t) = 2t$ is negative (i.e. $x(t)$ decreasing)

$$\underline{\text{area} = \frac{2^3 \cdot 3^4}{5}} \left(= \frac{648}{5}\right)$$