Name and surname: ________________________ Student ID: ___________________

	1a	1b	2	3a	3b	3c	3d	4a	4b	Sum
Points										

EXAM in Mathematical Modelling
June 27, 2025

For each of tasks justify all your answers.

1. a. **[5 points]** Let $A \in \mathbb{R}^{n\times r}$, $r < n$, be a matrix with pairwise orthogonal columns with lengths $\ell_1, \ell_2, \dots, \ell_r$. Express the Moore–Penrose inverse of the matrix A.

   b. **[7 points]** You are given data points $A(-\frac{\pi}{2}, 1)$ $B(0, 2)$, $C(\frac{\pi}{3}, 3)$ and $D(\frac{2\pi}{3}, 1)$. Find values for parameters $a, b \in \mathbb{R}$, such that the function

$$f(x) = a \sin x + b \cos x$$

   will be the best fit in the sense of the linear least squares method for these data points.

2. **[12 points]** A company is designing a small battery grid made up of three battery cells. Each cell has a voltage potential x_i (in volts), where $i = 1, 2, 3$. The cells are connected in a loop, and the voltage drops across the resistive links between them depend on the nonlinear behavior of certain semiconducting elements. The system is subject to the following physical constraints:

$$\begin{aligned} x_1^2 + \sin(x_2) + x_3 &= 5 \\ x_2^2 + \cos(x_3) + x_1 &= 4 \\ e^{x_3} + x_1 x_2 &= 6 \end{aligned}$$

Start with the initial guess $\mathbf{x}^{(0)} = [0, 0, 0]^\top$ and do one step of Newton's method to find a crude approximation to the solution of the above system.

3. Consider the curve parametrized by

$$x(t) = 1 + 3t^2, \quad y(t) = 2t^3, \quad \text{for } -2 \leq t \leq 2.$$

a. **[3 points]** Prove that the curve is mirror-symmetric with respect to the x-axis.

b. **[3 points]** Find the equation of the line tangent to the curve at the point $(4, 2)$.

c. **[4 points]** Find the area of the figure enclosed by the curve, and the lines $x = 1$ and $y = 2$.

d. **[4 points]** Find the arc length of the given curve.

4. A cup of tea is poured and has an initial temperature of 90°C. It is placed in a room where the air temperature is a constant 20°C. After 25 minutes, the tea has cooled down to 70°C.

   The rate of change of the tea temperature is proportional to the difference between the tea temperature and the room temperature. Assume that the room temperature remains constant.

   Let $T(t)$ represent the temperature of the tea at time t (in minutes) after it was placed in the room.

   a. **[4 points]** Write a differential equation that describes how T changes as a function of t.

   b. **[8 points]** Using your model, determine when the tea will cool down to 40°C (which is considered a good temperature for drinking).