Name and surname: ______________________ Student ID: __________________

	Problem 1	Problem 2	Sum
Points			

Midterm EXAM in Mathematical Modelling
March 11, 2026

For each of the tasks 1.-2. show your work and justify all your answers.

5,5 pts

1. Let $A \in \mathbb{R}^{m \times n}$ be a matrix with rows $\vec{a}_1^\top, \ldots, \vec{a}_m^\top$, which are pairwise orthogonal with nonzero lengths $s_1, \ldots, s_m \in \mathbb{R}$, i.e., $s_i = \|\vec{a}_i\|$ for all i.

   Express the Moore–Penrose inverse A^+ in terms of $\vec{a}_1, \ldots, \vec{a}_m \in \mathbb{R}^n$ and $s_1, \ldots, s_m$.

1,5

$A = \begin{bmatrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{bmatrix}$ with orthogonal $\neq \vec{0}$ rows = lin. indep.

$\Rightarrow \operatorname{rank}(A) = m$

$\Rightarrow$ A has full row rank

1

1

$s_i \neq 0$

$$\Rightarrow A^+ = A^T (AA^T)^{-1} =$$

$$= \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_m^T \end{bmatrix}^T \begin{bmatrix} s_1^2 & & 0 \\ & \ddots & \\ 0 & & s_m^2 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_m \end{bmatrix} \begin{bmatrix} \frac{1}{s_1^2} & & \\ & \ddots & \\ & & \frac{1}{s_m^2} \end{bmatrix} =$$

1

$$= \begin{bmatrix} \frac{1}{s_1^2}\vec{a}_1 & \frac{1}{s_2^2}\vec{a}_2 & \cdots & \frac{1}{s_m^2}\vec{a}_m \end{bmatrix}$$

1

(7pts) 2. A sports scientist studies how an athlete's competition performance score y depends on three measurable training factors:

- Training intensity $I(t)$: daily training load (unit: hours per day),
- Recovery quality $R(t)$: regeneration score (dimensionless index based on sleep and physiological markers),
- Nutrition support $N(t)$: standardized nutrition index (dimensionless score).

She assumes that the performance score can be modeled by a linear function

$$f(t) = \alpha_1 I(t) + \alpha_2 R(t) + \alpha_3 N(t),$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ are unknown parameters describing the influence of each factor.

In the pre-Olympic training camp she measured $(I_1, R_1, N_1) = (1, 1, 2)$ and $y_1 = 48$ on day 1, and $(I_2, R_2, N_2) = (2, -4, 1)$ and $y_2 = 21$ on day 2.

Although she knows it is too early, she determines the parameters $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that the function $f(t)$ provides the best linear fit to the observed data with the smallest possible value for $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$. What performance score would she predict for day 3 if $(I_3, R_3, N_3) = (1, 2, 2)$?

Formalization: 3 points

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 48$$
$$2\alpha_1 - 4\alpha_2 + \alpha_3 = 21$$

$$\rightsquigarrow \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & -4 & 1 \end{bmatrix}}_{= A} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{= \vec{x}} = \underbrace{\begin{bmatrix} 48 \\ 21 \end{bmatrix}}_{= \vec{b}} \quad \leftarrow 1$$

We're looking for the solution of $A\vec{x} = \vec{b}$ with the smallest $\|\vec{x}\|^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$. $\leftarrow 1$

Solution to this problem: $A^+\vec{b}$. $\leftarrow 1$

Computation of parameters: 3 points

A has orthogonal rows $\vec{a}_1, \vec{a}_2$ with $\|\vec{a}_1\| = \sqrt{6}$ and $\|\vec{a}_2\| = \sqrt{21}$.

By Problem 1:

$$A^+ = \begin{bmatrix} \frac{1}{6}\vec{a}_1 & \frac{1}{21}\vec{a}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{2}{21} \\ \frac{1}{6} & -\frac{4}{21} \\ \frac{1}{3} & \frac{1}{21} \end{bmatrix} \leftarrow 2 \text{ (in any way)} \quad \leftarrow (*)$$

$$A^+\vec{b} = \begin{bmatrix} 8+2 \\ 8-4 \\ 16+1 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 17 \end{bmatrix} \Rightarrow \alpha_1 = 10, \alpha_2 = 4, \alpha_3 = 17$$

1

Prediction: 1 point

Day 3: $10 \cdot 1 + 4 \cdot 2 + 17 \cdot 2 = 10 + 8 + 34 = 52$ $\leftarrow 1$

⊛ Alternative for A^+:

$$A^+ = A^T (AA^T)^{-1} = A^T \cdot \begin{bmatrix} 6 & 0 \\ 0 & 21 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{21} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{2}{21} \\ \frac{1}{6} & -\frac{4}{21} \\ \frac{1}{3} & \frac{1}{21} \end{bmatrix}$$

⊛ or with SVD for A.