Name and surname: ______________________ Student ID: __________________

	Problem 1	Problem 2	Sum
Points			

Midterm EXAM in Mathematical Modelling
March 26, 2025

For each of tasks 1.-2. show your work and justify all your answers.

1. **(4 points)** A function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = \ln(x^2 + 2y) + \sin(xy).$$

Find the equation of the plane, tangent to the surface $z = f(x,y)$ through point $(1, 0, f(1,0))$.

$f(x,y,z) = \ln(x^2 + 2y) + \sin(xy)$ $\quad$ $f(1,0) = \ln(1) + \sin(0) = 0$

$(\text{grad } f)(x,y,z) = \left(\frac{2x}{x^2+2y} + y\cos(xy) \, , \, \frac{2}{x^2+2y} + x\cos(xy) \right)$ $\quad$ 1

$(\text{grad } f)(1,0) = 1 \left(\frac{2}{1} + 0, \frac{2}{1} + 1 \right) = (2,3)$ $\quad$ 1

$z = 0 + [2, 3] \begin{bmatrix} x-1 \\ y-0 \end{bmatrix} = 2x - 2 + 3y$

1

$2x + 3y - z = 2$

2. **(8.5 points)** Let $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\vec{F}(x,y) = \begin{bmatrix} \ln(x^2+2y) \\ \sin(xy) \\ \sqrt{xy^2+1} \end{bmatrix}.$$

Use the initial approximation $(x_0, y_0) = (0,1)$ and perform one step of (Gauss-)Newton's method to find an approximate solution to

$$\vec{F}(x,y) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$\vec{G}(x,y,z) := \begin{bmatrix} \ln(x^2+2y) \\ \sin(xy) - 1 \\ \sqrt{xy^2+1} - 2 \end{bmatrix} \quad 1$$

Want to find solutions to $\vec{G}(x,y) = \vec{0}$.

$$J_{\vec{G}} = \begin{bmatrix} \frac{2x}{x^2+2y} & \frac{2}{x^2+2y} \\ y\cos(xy) & x\cos(xy) \\ \frac{y^2}{2\sqrt{xy^2+1}} & \frac{2xy}{2\sqrt{xy^2+1}} \end{bmatrix} \quad 1$$

$$J_{\vec{G}}(0,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \quad 1$$

$$\vec{G}(0,1) = \begin{bmatrix} \ln(2) \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \ln(2) \\ -1 \\ -1 \end{bmatrix} \quad 1$$

$$\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_1 = \vec{x}_0 - \underbrace{\underbrace{J_{\vec{G}}(\vec{x}_0)}_{A} \setminus \vec{G}(\vec{x}_0)}_{1}$$

overdetermined system

A has full column rk

$$A^T A = \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^+ = (A^T A)^{-1} A^T = \begin{bmatrix} 0 & \frac{4}{5} & \frac{2}{5} \\ 1 & 0 & 0 \end{bmatrix} \quad 1$$

$$\Rightarrow A^+ b = \begin{bmatrix} -\frac{6}{5} \\ \ln 2 \end{bmatrix} \quad 1$$

$$= \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{6}{5} \\ \ln 2 \end{bmatrix}}_{1} = \underbrace{\begin{bmatrix} \frac{6}{5} \\ 1 - \ln 2 \end{bmatrix}}_{0,5}$$