Name and surname: ____________________ Student ID: ________________

	1a.	1b.	1c.	1d.	2	Sum
Points						

Midterm EXAM in Mathematical Modelling
April 8, 2026

For each of tasks justify all your answers.

1. Consider the plane curve $\mathcal{C}$ parametrized by

$$x(t) = t^3 - 3t, \qquad y(t) = t^2 - 1, \qquad t \in \mathbb{R}.$$

$= t(t^2-3)$

$\vec{F}(t) = \begin{bmatrix} t^3-3t \\ t^2-1 \end{bmatrix}$

a. (1 point) Show that the curve $\mathcal{C}$ is symmetric over the y-axis.

$x(-t) = -t^3 + 3t = -x(t)$
$y(-t) = t^2 - 1 = y(t)$
$\Rightarrow$ symmetric over y-axis

1

b. (1 point) Prove that $\mathcal{C}$ has a self-intersection.

$\vec{F}(t) = \vec{F}(s)$? $\Leftrightarrow$ $t^3 - 3t = s^3 - 3s$
$t^2 - 1 = s^2 - 1$ $\Rightarrow$ $t = s$ or $t = -s$
$t = s$ not a selfintersection

$-s^3 + 3s = s^3 - 3s$
$2s^3 - 6s = 0$ $\quad /:2 \; /:s$ ($s \neq 0$) ($s = 0$ would not be a self-int.)
$s^2 - 3 = 0$
$s = \pm\sqrt{3}$
$\Downarrow$
$t = \mp\sqrt{3}$

$\vec{F}(\sqrt{3}) = (0, 2) = \vec{F}(-\sqrt{3})$ is the only self-intersection

1

c. (4 points) For the curve $\mathcal{C}$ compute the following and write the results in the table.

notation for part

	t	point $(x(t), y(t))$	
1) Intersections with x-axis	1 -1	$(-2,0)$ P_1 $(2,0)$ P_2	1
2) Intersections with y-axis	$-\sqrt{3}$ 0 $\sqrt{3}$	$(0,2)$ P_3 ⇐ self-intersec. $(0,-1)$ P_4 $(0,2)$ P_3	1
3) Points with horizontal tangents	0	$(0,-1)$ P_4	1
4) Slopes of tangents in self-intersections	$-\sqrt{3}$ $\sqrt{3}$	$-\frac{\sqrt{3}}{3}$ $\approx -\frac{2}{3}$ $\frac{\sqrt{3}}{3}$ $\approx \frac{2}{3}$	1

1) $y(t) = 0$
$t^2 - 1 = 0$
$t = \pm 1$

2) $x(t) = 0$
$t(t-\sqrt{3})(t+\sqrt{3}) = 0$
$t = 0 \lor t = \pm\sqrt{3}$

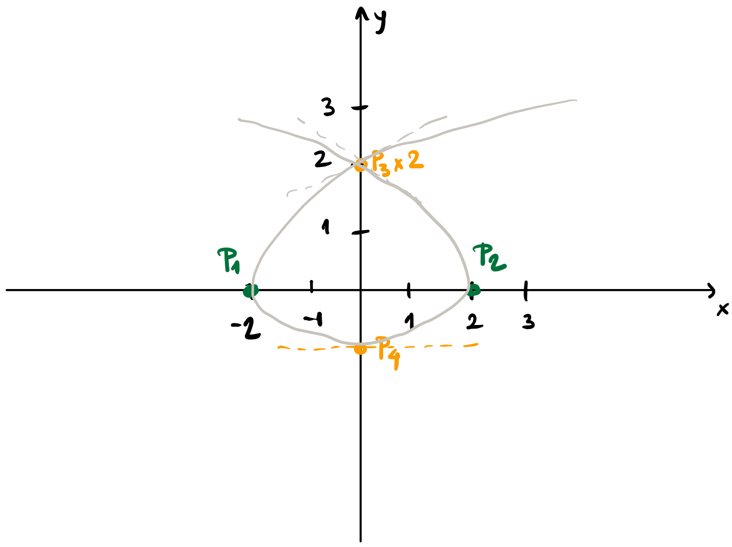
3) $y'(t) = 0$
$2t = 0$
$t = 0$
$x'(t) = 3t^2 - 3$
$x'(0) = -3 \neq 0$

4) $\vec{F}'(t) = \begin{bmatrix} 3t^2 - 3 \\ 2t \end{bmatrix}$

$$\frac{y'(t)}{x'(t)} = \frac{2t}{3t^2 - 3} \xrightarrow{t \to \sqrt{3}} \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

$$\xrightarrow{t \to -\sqrt{3}} \frac{-2\sqrt{3}}{6} = -\frac{\sqrt{3}}{3}$$

d. (1.5 points) Sketch the curve $\mathcal{C}$ using the information provided in the table from part c. (No arguments needed.)



2. **(5 points)** A company is designing a spiral slide inside a tall observation tower. The center-line of the slide follows a three-dimensional curve given by

$$x(t) = 2\cos(2t), \quad y(t) = 2\sin(2t), \quad z(t) = 5 - t,$$

where $t \in [0, 5]$, and all distances are measured in meters. To estimate the amount of material needed for the slide's support structure, engineers must determine the total length of the path along which the slide is built. Compute the total arc length of the slide from the top ($t = 0$) to the bottom ($t = 5$).

$$\vec{r}(t) = \begin{bmatrix} 2\cos^3(2t) \\ 2\sin^3(2t) \\ 5-t \end{bmatrix} \qquad \vec{r}\,'(t) = \begin{bmatrix} -4\sin(2t) \\ 4\cos(2t) \\ -1 \end{bmatrix} \quad 1$$

$$\|\vec{r}\,'(t)\|^2 = 16\sin^2(2t) + 16\cos^2(2t) + 1 = 17$$

$$\|\vec{r}\,'(t)\| = \sqrt{17} \qquad 1$$

$$\ell = \int_0^5 \|\vec{r}\,'(t)\|\,dt = \underbrace{\int_0^5 \sqrt{17}\,dt}_{1} = \sqrt{17}\; t\Big|_{t=0}^{5} = 5\sqrt{17}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \qquad\qquad 1$$