Name and surname: ______________________ Student ID: ____________________

	1	~~1b~~	2a	2b	2c	Sum
Points	8,5			4		

Midterm EXAM in Mathematical Modelling
May 6, 2026

For each of tasks justify all your answers.

1. **(8.5 points)** Find the solution $y = y(x)$ of the differential equation

$$y''' - 3y'' + 4y = e^{3x}$$

with initial condition $y(0) = \frac{1}{4}$, $y'(0) = \frac{3}{4}$ and $y''(0) = \frac{5}{4}$.

homogeneous part: $y''' - 3y'' + 4y = 0 \longrightarrow$ char. poly $t^3 - 3t^2 + 4 = (t+1)(t^2-4t+4) = (t+1)(t-2)^2$ (1)

3 pts

$\lambda_1 = -1,\ \lambda_2 = \lambda_3 = 2$ (1)

$$\Rightarrow \boxed{y_H(x) = C_1 e^{-x} + C_2 e^{2x} + C_3 x e^{2x}} \quad (1)$$

nonhom. part: guess one solution:

2 pts

$$\left.\begin{aligned} y(x) &= Ae^{3x} \\ y'(x) &= 3Ae^{3x} \\ y''(x) &= 9Ae^{3x} \\ y'''(x) &= 27Ae^{3x} \end{aligned}\right\} \xrightarrow{\text{into DE}} \cancel{27Ae^{3x}} - \cancel{27Ae^{3x}} + 4Ae^{3x} = e^{3x} \ /:e^{3x} \neq 0$$

(1)

$4A = 1$

$A = \frac{1}{4}$ (1)

$$\Rightarrow \boxed{y_P(x) = \frac{1}{4}e^{3x}}$$

$$\Rightarrow \boxed{y(x) = C_1 e^{-x} + C_2 e^{2x} + C_3 x e^{2x} + \frac{1}{4}e^{3x}} \leftarrow \text{general solution} \leftarrow \text{1 pt}$$

(1)

$y(0) = C_1 + C_2 + \frac{1}{4}$

$y'(x) = -C_1 e^{-x} + 2C_2 e^{2x} + C_3 e^{2x} + 2C_3 x e^{2x} + \frac{3}{4}e^{3x}$

$y'(0) = -C_1 + 2C_2 + C_3 + \frac{3}{4}$

$y''(x) = C_1 e^{-x} + 4C_2 e^{2x} + 2C_3 e^{2x} + 2C_3 e^{2x} + 4C_3 x e^{2x} + \frac{9}{4}e^{3x}$

$y''(0) = C_1 + 4C_2 + 4C_3 + \frac{9}{4}$

2pts for initial conditions

$C_1 + C_2 = 0$

$-C_1 + 2C_2 + C_3 = 0 \quad /\cdot(-4)] +$

$C_1 + 4C_2 + 4C_3 = -1$

$C_1 + C_2 = 0 \quad /\cdot 4] +$

$5C_1 - 4C_2 = -1$

$9C_1 = -1,\ C_1 = -\frac{1}{9}$

$C_2 = \frac{1}{9}$

(1)

$C_3 = C_1 - 2C_2 = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$

$$\Rightarrow \boxed{y(x) = -\frac{1}{9}e^{-x} + \frac{1}{9}e^{2x} - \frac{1}{3}xe^{2x} + \frac{1}{4}e^{3x}}$$

0,5 for final result

2. **(4 points)** Consider an ecosystem consisting of three populations: algae $A(t)$, shrimp $S(t)$, and fish $F(t)$.

The algae population at time t experiences a natural decline proportional to its current size $A(t)$. However, it is replenished by "nutrient recycling" from the shrimp population, increasing at a rate proportional to the shrimp population size.

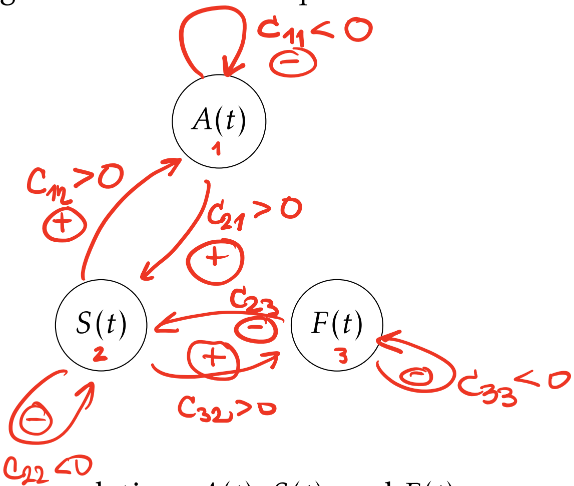
The shrimp population depends on algae as a food source, so it increases at a rate proportional to the algae population size. However, shrimp are eaten by fish at a rate proportional to the fish population size, and they also experience natural mortality proportional to their own size.

The fish population grows at a rate proportional to the shrimp population size, but it also experiences a natural death rate proportional to its own size.

1

A. Complete the interaction diagram below by adding arrows and self-loops to show how the populations affect one another.

- Draw arrows between nodes to represent interactions.
- Add self-loops to represent natural growth or decay.
- Label each interaction with the corresponding coefficient c_{ij}.
- Assign a "+" or "−" sign to indicate whether the interaction increases or decreases the target population.



B. Write down differential equation(s) that models the populations $A(t)$, $S(t)$, and $F(t)$.

1

$$A'(t) = c_{11} A(t) + c_{12} S(t)$$
$$S'(t) = c_{21} A(t) + c_{22} S(t) + c_{23} F(t)$$
$$F'(t) = c_{32} S(t) + c_{33} F(t)$$

(Note: also ok if you wrote − instead of +, since the coefficients need not be of determined sign.)

C. Describe the steps you would take to solve the problem in part B. Do not compute the solution.

2

1) $$\begin{bmatrix} A' \\ S' \\ F' \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{bmatrix}}_{=: C} \begin{bmatrix} A \\ S \\ F \end{bmatrix}$$

2) Compute eigenvalues and eigenvectors for C

3) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ lin indep., then

$$\begin{bmatrix} A \\ S \\ F \end{bmatrix} = \sum_{i=1}^{3} D_i e^{\lambda_i x} \vec{v}_i$$