Geodesics

Using the method of Christoffel symbols

Background

In Euclidean space (e.g. $\mathbb{R}^2$ and $\mathbb{R}^3$), the shortest path between two points is a line. However, in curved spaces (e.g. surfaces in $\mathbb{R}^3$ such as a sphere or a torus), lines in general do not exist, and the straightest paths between two points are curves called *geodesics*. Sometimes geodesics coincide with shortest paths, but in general these are different concepts. Geodesics are calculated as solution curves of a system of differential equations. This system can be obtained quite easily for some special cases of implicit surfaces in $\mathbb{R}^3$, but the task of this project is to derive it by a process that works for any surface that looks locally, in the neighbourhood of any point, like the usual Euclidean space $\mathbb{R}^2$. Such surfaces are called 2–dimensional Riemannian manifolds.

System of differential equations determining geodesics. Let $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^3$ be a map that locally parametrizes a surface. It maps 2–dimensional coordinates (u, v) into 3–dimensional coordinates (X, Y, Z):

$$\vec{f}(u, v) = (X(u, v), Y(u, v), Z(u, v)).$$

Example 1. Here is an example of the parametrization of a sphere in $\mathbb{R}^3$:

$$\begin{aligned} X &= \cos(v)\sin(u), \\ Y &= \sin(v)\sin(u), \\ Z &= \cos(u), \end{aligned} \tag{1}$$

where $u, v \in \mathbb{R}$. Note that for bijective correspondence the parameters u and v must lie only on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[0, 2\pi)$, respectively. ■

For the sake of simplicity let $u^1 = u$ and $u^2 = v$. We define the *Christoffel symbols* as

$$\Gamma^k_{ij} = \sum_{\ell=1}^{2} \left\langle \frac{\partial^2 \vec{f}}{\partial u^i \partial u^j}, \frac{\partial \vec{f}}{\partial u^\ell} \right\rangle \mathfrak{g}^{\ell k}, \qquad i, j, k = 1, 2, \tag{2}$$

where $\frac{\partial^2 \vec{f}}{\partial u^i \partial u^j}$ is the second partial derivative of $\vec{f}$ w.r.t. u^i and u^j, $\frac{\partial \vec{f}}{\partial u^\ell}$ is the partial derivative of $\vec{f}$ w.r.t. u^ℓ, $\langle \cdot, \cdot \rangle$ stands for the usual inner product of vectors and $\mathfrak{g}^{\ell k}$ is the element in the ℓ–th row and k–th column of the inverse of the *metric tensor* G. As the name suggests G is a tensor in general, but for our case it is a matrix defined as:

$$G(u^1, u^2) = \begin{bmatrix} \left\langle \frac{\partial \vec{f}}{\partial u^1}, \frac{\partial \vec{f}}{\partial u^1} \right\rangle & \left\langle \frac{\partial \vec{f}}{\partial u^1}, \frac{\partial \vec{f}}{\partial u^2} \right\rangle \\ \left\langle \frac{\partial \vec{f}}{\partial u^2}, \frac{\partial \vec{f}}{\partial u^1} \right\rangle & \left\langle \frac{\partial \vec{f}}{\partial u^2}, \frac{\partial \vec{f}}{\partial u^2} \right\rangle \end{bmatrix}. \tag{3}$$

Once we obtain the Christoffel symbols the system of two differential equations (DEs) determining geodesic curves is the following:

$$\frac{d^2 u^k}{dt^2} + \sum_{i,j=1}^{2} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0, \qquad k = 1, 2, \tag{4}$$

where $t > 0$ is the parameter. For a comprehensive derivation of Christoffel symbols and a system of DEs see [1]. This system may seem complicated at first glance, but for some surfaces we get quite simple equations. In general, however, this system is difficult to solve analytically, so the appropriate method is to solve it numerically. The Euler method can work well for some special cases, but in general it is not stable and causes large numerical errors. A better way to solve such a system is to use an adaptive method such as the Dormand-Prince 5 (DOPRI5) method. Note that you must first transform the given system of DEs into a system of first order DEs in order to apply one of these methods.

A procedure to derive a system of DEs determining the geodesics on a surface parametrized by $\vec{f}$.

(1) Calculate the inverse of the metric tensor 3.
(2) Calculate the Christoffel symbols by 2.
(3) Obtain the system of DEs by 4.

Solving the system of DEs obtained.

(1) Pick a starting point $P = \vec{f}(u_0, v_0) \in \mathbb{R}^3$ on the surface and a starting direction $(d_1, d_2) \in \mathbb{R}^2$.
(2) Perform a step of a chosen numerical method for solving the system 4 and obtain a new point on the surface.
(3) Repeat step 2 for the chosen amount of iterations depending on the desired length of move.

TASK

(1) Derive the geodesic equations for a plane parametrized by $\vec{f}(u, v) = \vec{p} + u\vec{a} + v\vec{b}$, where $\vec{p} \in \mathbb{R}^3$ is a point and $\vec{a}, \vec{b} \in \mathbb{R}^3$ are linearly independent vectors. You should obtain the equation of a line.
(2) Derive the geodesic equations for a sphere parametrized by 1.
(3) Derive the geodesic equations for a torus parametrized by

$$\begin{aligned} X(u, v) &= (R + r\cos u)\cos v \\ Y(u, v) &= (R + r\cos u)\sin v \\ Z(u, v) &= r\sin u, \end{aligned}$$

where R is the big radius, r is the small radius of a torus,$u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $v \in [0, 2\pi)$

(4) Plot a few geodesics for the sphere and the torus.

REFERENCES

[1] eigenchris, *Tensor Calculus 15: Geodesics and Christoffel Symbols (extrinsic geometry)* `https://www.youtube.com/watch?v=1CuTNveXJRc`, Accessed 16. 2. 2024