Drugi kolokvij (8.1.2025)

* REŠITVE *

1. naloga **(30 točk);** Dana je funkcija f s predpisom $f(x) = \log(\frac{x^2+1}{x^2+2})$.

a) **(15 točk)** Poiščite stacionarne točke funkcije f. V katerih od teh točk so lokalni minimumi oz. lokalni maksimumi?

$$f(x) = \log\left(\frac{x^2+1}{x^2+2}\right)$$

$$f'(x) = \frac{1}{\frac{x^2+1}{x^2+2}} \cdot \frac{2x(x^2+2) - (x^2+1)2x}{(x^2+2)^2} =$$

$$= \frac{2x(\cancel{x^2}+2-\cancel{x^2}-1)\cancel{(x^2+2)}}{(x^2+1)(x^2+2)^{\cancel{2}}} =$$

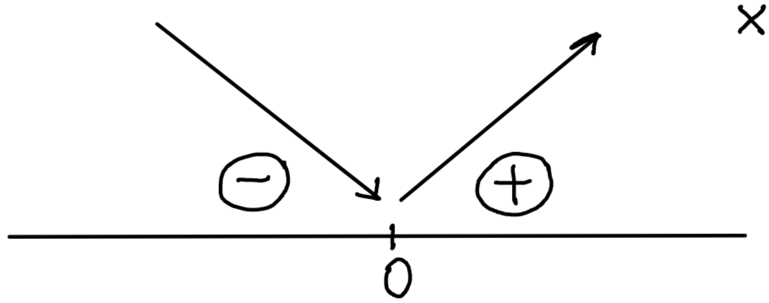
$$= \frac{2x}{(x^2+1)(x^2+2)}$$

stacionarne točke: $f'(x) = 0$

$$2x = 0$$

$$x = 0$$

f' :



Točka $T(0, \log(\frac{1}{2}))$ je lokalni minimum.

b) **(5 točk)** Na katerih intervalih je funkcija f naraščajoča oz. padajoča?

Funkcija narašča na $(0,\infty)$ in pada na $(-\infty,0)$.

c) **(10 točk)** Poiščite tangento na graf funkcije f v točki $T(1, f(1))$.

Enačba tangente na graf f v točki $(1, f(1)) = (1, \log(\frac{2}{3}))$:

$$y - y_0 = f'(x_0)(x - x_0)$$

$$y - \log\left(\frac{2}{3}\right) = \frac{1}{3}(x-1)$$

$$\underline{\underline{y = \frac{1}{3}x - \frac{1}{3} + \log\left(\frac{2}{3}\right)}}$$

$$f'(x_0) = f'(1) = \frac{2\cdot 1}{2\cdot 3} = \frac{1}{3}$$

2. naloga **(35 točk)**;

a) **(15 točk)** Izračunajte določeni integral $\int\limits_0^{\frac{\pi}{4}} \frac{x^2}{x^2+1}\, dx$.

$$\int_0^{\frac{\pi}{4}} \frac{x^2}{x^2+1}\, dx$$

$$\left(\begin{array}{l} x^2 : (x^2+1) = 1 \\ \underline{-x^2 \pm 1} \\ \quad -1 \end{array} \qquad \frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1} \right)$$

$$\int_0^{\frac{\pi}{4}} \frac{x^2}{x^2+1}\,dx = \int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{x^2+1}\right)dx =$$

$$= \left(x - \arctan x\right)\Big|_0^{\frac{\pi}{4}} =$$

$$= \left(\frac{\pi}{4} - \arctan\left(\frac{\pi}{4}\right)\right) - \left(0 - \arctan 0\right) =$$

$$= \frac{\pi}{4} - 1 = \underline{\underline{\frac{\pi - 4}{4}}}$$

b) **(20 točk)** Izračunajte nedoločeni integral $\int x \arctan(x)\, dx$.

$$\int x \cdot \arctan x \, dx =$$

PER PARTES:

$$\left[\begin{array}{ll} u = \arctan x & du = \frac{1}{1+x^2}\,dx \\ dv = x\,dx & v = \frac{x^2}{2} \end{array}\right]$$

$$= \frac{x^2}{2} \cdot \arctan x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2}\,dx =$$

$$= \frac{x^2}{2} \cdot \arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx =$$

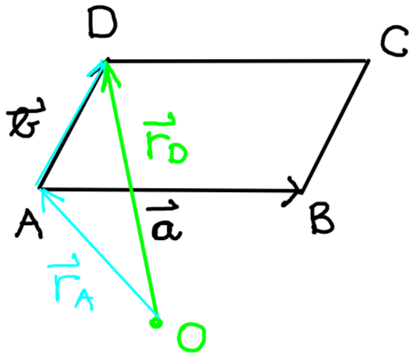
$$\overset{(a)}{=} \frac{x^2}{2} \cdot \arctan x - \frac{1}{2}\left(x - \arctan x\right) + C =$$

$$= \frac{x^2}{2} \cdot \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C =$$

$$= \frac{1}{2}\arctan x \left(x^2+1\right) - \frac{1}{2}x + C$$

3. naloga **(35 točk);** Podana imamo tri izmed oglišč paralelograma $ABCD$: $A(-1,0,1)$, $B(2,1,3)$ in $C(3,3,5)$. Pri tem je AC diagonala paralelograma $ABCD$.

a) **(10 točk)** Določite koordinate točke D.



$$\vec{a} = \overrightarrow{AB} = \vec{r}_B - \vec{r}_A = \begin{bmatrix}3\\1\\2\end{bmatrix}$$

$$\vec{b} = \overrightarrow{BC} = \vec{r}_C - \vec{r}_B = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

$$\vec{r}_D = \vec{r}_A + \vec{b} = \begin{bmatrix}0\\2\\3\end{bmatrix}$$

$D(0,2,3)$

b) **(10 točk)** Izračunajte dolžini obeh diagonal v paralelogramu $ABCD$.

$$\overrightarrow{AC} = \vec{r}_C - \vec{r}_A = \begin{bmatrix}4\\3\\4\end{bmatrix} \qquad |\overrightarrow{AC}| = \sqrt{4^2+3^2+4^2} = \sqrt{41}$$

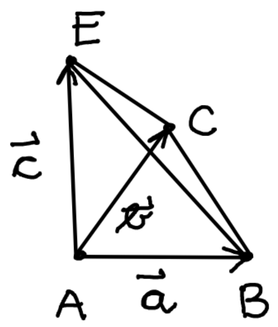
$$\overrightarrow{BD} = \vec{r}_D - \vec{r}_B = \begin{bmatrix}-2\\1\\0\end{bmatrix} \qquad |\overrightarrow{BD}| = \sqrt{(-2)^2+1^2+0^2} = \sqrt{5}$$

c) **(10 točk)** Izračunajte ploščino paralelograma $ABCD$.

$$\vec{a} \times \vec{b} = \begin{bmatrix}3\\1\\2\end{bmatrix} \times \begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}-2\\-4\\5\end{bmatrix}$$

$$S_{\square} = |\vec{a} \times \vec{b}| = \sqrt{(-2)^2+(-4)^2+5^2} = \sqrt{45} = 3\sqrt{5}$$

c) **(5 točk)** Naj bo E točka s koordinatimi $E(5,0,7)$. Izračunajte prostornino piramide $ABCE$.



$$\vec{a} = \overrightarrow{AB} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{c} = \overrightarrow{AE} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

$$\vec{b} = \overrightarrow{AC} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$(\vec{a}, \vec{b}, \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 18 \\ 0 \\ -18 \end{bmatrix} = 3 \cdot 18 + 1 \cdot 0 + 2 \cdot (-18)$$

$$= 18$$

$$\vec{b} \times \vec{c} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ -18 \end{bmatrix}$$

$$V = \frac{|(a,b,c)|}{6} = \frac{18}{6} = 3$$