· Name and surname: ________________________ Vpisna številka: ___________________

1. Exam for OMA, 23.01.2020

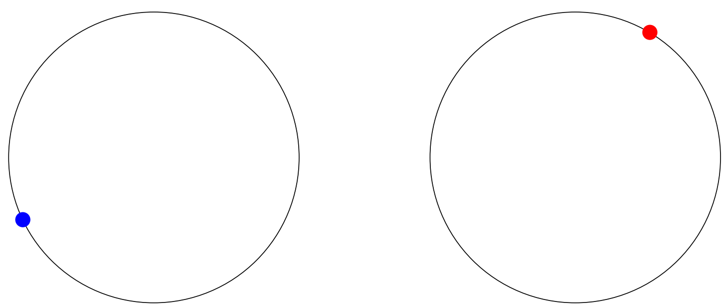
- Time for exam: **45 minut**
- If you do not have enough space to write an answer on the exam paper, mark this and continue on the additional paper.
- For passing grade you have to achieve at least 50% of all points and also 30% of the points at each task, i.e., 1.5 points out of 5 points. The number in the bracket [·] tells you how many points you get for a correct solution to the question.
- Any attempt of copying the solution of someone else, talking, using notes or electronic equipement is **strictly** forbidden.

1. **[5 points] Mathematical induction and number sets**

(a) [**1**] Let $T(n)$ be a statement about the natural number $n \in \mathbb{N}$. We are given that $T(3)$ is true and that $T(n)$ being true implies $T(n+4)$ is true. What can we say about the truth of $T(2020)$? Explain your answer.

(b) [**2**] Explain what is the n-th root of a complex number $a \in \mathbb{C}$. Write down also explicit formulas for all n-th roots of the number $a \in \mathbb{C}$.

(c) [**2**] Let us have $n_1 = 2$ and $n_2 = 6$. On the left image we see one of the n_1-th roots of some complex number, while on the right one of the n_2-th root. On both images draw points for all the other roots. The procedure must be clearly seen. Use the fact that the center of both circles is the point $(0,0)$.



2. **[5 points] Sequences and series**

(a) [**1**] Write down the definition of a supremum of a sequence $\{a_n\}_{n\in\mathbb{N}}$, $a_n \in \mathbb{R}$.

(b) **[1]** Write down the convergence theorem for monotone sequences.

(c) **[3]** Determine which of the sequences below are convergent. Justify both answers. You can use all facts about those sequences and series which we studied on lectures.

i. **[1]** $b_0 = 0$, $b_n = b_{n-1} + \frac{1}{n}$ for $n \geq 1$.

ii. **[2]** $c_0 = 0$, $c_n = c_{n-1} + (-1)^n \left(e - \left(1 + \frac{1}{n}\right)^n\right)$ for $n \geq 1$.

3. **[5 points] Functions**

(a) **[2]** Write down $\epsilon - \delta$ definition of a continuous function $f : \mathbb{R} \to \mathbb{R}$ in one variable in the point $x_0 \in \mathbb{R}$.

(b) **[3]** Let $f : [0,1] \to \mathbb{R}$ be a *continuous* function, which satisfies

$$f(0) = 1, \quad \lim_{x \nearrow \frac{1}{4}} f(x) = -1, \quad \lim_{x \searrow \frac{1}{2}} f(x) = 4, \quad \lim_{x \to \frac{3}{5}} f(x) = 5, \quad f(1) = -2,$$

and a linear function $g : [-2, 5] \to [0, 14]$, given by the rule $g(x) = 2(x+2)$. Below there are few statements about functions f, g. Circle P, if the statement is correct and N otherwise.

Caution: for each correct answer you get 0.5 point, for each wrong one you *lose* 0.5 point. If you do not answer, you get 0 points. The total number of points you get for this part cannot be negative.

i. **[0.5]** f is onto. P / N

ii. **[0.5]** It holds that $\lim_{x \searrow \frac{1}{4}} f(x) = -1$. P / N

iii. **[0.5]** f has at least 3 zeores. P / N

iv. **[0.5]** It holds that $\lim_{x \to \frac{1}{2}} (g \circ f)(x) = 12$. P / N

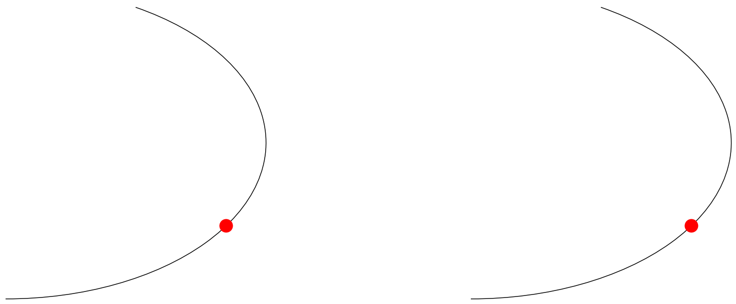
v. **[0.5]** The composite function $g \circ f$ is onto. P / N

vi. **[0.5]** The inverse of the function g is well-defined and equal to $g^{-1}(x) = \frac{x}{2} - 2$. P / N

4. [**5 točk**] **Derivative**

(a) [**1**] Write down the definition of the directional derivative of a differentiable function $f(x, y)$ in two variables in the direction of the vector $\vec{e} = (e_1, e_2)$ in the point (x_0, y_0).

(b) [**1**] The next sketches are parts of some level curve of a function of two variables. On the **left** sketch draw both directional vectors in the direction of the fastest change of the function value in the given point, while on the **right** both directional vectors in the direction of no change of the function value in the given point.



(c) [**3**] Let f be twice continuously differentiable function of two variables. For each of the points $P, Q, R \in \mathbb{R}^2$ determine and **justify** which are local extremes. Also classify all local extremes.

i. [**1**] $f_x(P) = f_y(P) = 0$, $f_{xx}(P) = 3$, $f_{xy}(P) = -1$, $f_{yy}(P) = 1$.

ii. [**1**] $f_x(R) = 0$, $f_y(R) = -1$, $f_{xx}(R) = 3$, $f_{xy}(R) = 0$, $f_{yy}(R) = 2$.

iii. [**1**] $f_x(Q) = f_y(Q) = 0$, $f_{xx}(Q) = 3$, $f_{xy}(Q) = 2$, $f_{yy}(Q) = 1$.

5. [**5 točk**] **Integral**

(a) [**1**] Write down the definition of indefinite integral of the function $f : (a, b) \to \mathbb{R}$.

(b) [**1**] Let F, G be indefinite integrals of the function $f : \mathbb{R} \to \mathbb{R}$ where $F(1) = 0$, $G(1) = 2$ and $F(5) = 4$. What is the value of $G(5)$? Justify your answer (only a number does not suffice).

(c) [**3**] Let $f(x) = \int_0^x e^{-t^2} dt$ be a function. Use the fact that $\lim_{x \to \infty} f(x) = \frac{\sqrt{\pi}}{2}$.

Below there are few statements about the function f. Circle P, if the statement is correct and N otherwise.

Caution: for each correct answer you get 0.5 point, for each wrong one you *lose* 0.5 point. If you do not answer, you get 0 points. The total number of points you get for this part cannot be negative.

i. [**0.5**] f is odd. P / N

ii. [**0.5**] f is increasing. P / N

iii. [**0.5**] The point $x = 0$ is a stationary point of f. P / N

iv. [**0.5**] f has exactly one cusp. P / N

v. [**0.5**] f is convex on $[0, \infty)$. P / N

vi. [**0.5**] f has local extremes, but no global ones. P / N

6. [**5 točk**] **Diferential equations**

(a) [**1**] Write down the definition of a differential equation with separable variables of order 1.

(b) [**4**] Solve the differential equation $y'(x) + 2y(x) = 2$ with the initial condition $y(0) = 2$.