Izpit iz Osnov matematične analize

- Time limit: **45 minutes**

- All the results should be written on this paper. Additional calculations and arguments have to be included.

- Exam is closed book, mobile phones and other devices are **strictly** prohibited.

1. **[20 points] Complex numbers**

   (a) Write numbers $z_1 = -1$, $z_2 = -i$ and $z_3 = 1 - i$ in polar form and plot them in the complex plane.

   (b) Describe and plot the set A of complex numbers with the property that their distance from z_1 is at most 2, and the set B of complex numbers equidistant from z_1 and z_2.

   (c) Draw the images of sets A and B with the map $z \mapsto z - 1$.

   (d) Draw the images of sets A and B with the map $z \mapsto z(1 - i)$.

2. **[15 points] Sequences and series**

   (a) A sequence (a_n) is increasing if _____.

   (b) Find the necessary and sufficient condition for the convergence of an increasing sequence: an increasing sequence (a_n) is convergent if and only if

   _____

   (c) For each of the following sequences determine if it is increasing or convergent and find the limit (if it exists):

   $$a_n = \frac{n-1}{n+1}$$

   $$b_n = -n^2 + 20n$$

   $$c_n = (1 + \tfrac{1}{n})^n$$

3. **[15 points] Functions**

   (a) A function f is injective if

   _____

   For each of the given functions determine if it is injective. If it is, find its inverse function. If not, give a reason why.

   (b) $f(x) = \log(x^2 - 1)$

   (c) $h(x) = \log(2x - 1)$

4. **[20 points] Derivatives**

(a) What is a gradient of a function of two variables $f(x, y)$?

(b) In what direction does the vector $\operatorname{grad} f(x, y)$ point when it is non-zero??

(c) Write down the gradient of the function $f(x, y) = \sqrt{x^2 + y^2}$.

(d) Draw the level curve of the function $f(x, y) = \sqrt{x^2 + y^2}$ through the point $(1, 1)$ and find the directional derivative at this point in the direction, tangent to the level line.

5. **[20 points] Extremes** A function $f(x)$ is infinitely differentiable (smooth).

(a) What is a stationary point of $f(x)$?

(b) How can we use the first derivative to determine if a stationary point is a local extreme and to determine the type of the extreme?

(c) If $f'(x_0) = f''(x_0) = 0$ and $f^{(4)}(x_0) < 0$, what condition on $f'''(x_0)$ ensures us that x_0 is a local extreme? Is it a minimum or a maximum?

(d) Does the function $f(x) = \int_0^x t(e^{-t} - 1)\, dt$ have a local extreme at $x_0 = 0$? Why? If yes, is it a minimum or a maximum?

6. **[20 points] Integrals**

(a) What is the integral sum $f(x)$ on the interval $[a, b]$?

$$S_n =$$

Write down the definition of the definite integral

$$\int_a^b f(x)\, dx =$$

The integral $\int_0^2 \sqrt{4 - x^2}\, dx$ represents the area of a planar figure. Draw this figure. What is the value of this integral?