Third <u>Theoretical</u> Exam for OMA, 07.09.2020

- Time limit: **45 minutes**
- For a passing grade you have to achieve at least 50% of all points. The number in the bracket $[\cdot]$ tells you how many points you get for a correct solution to the question.
- Any attempt of copying the solution of someone else, talking, using electronic equipment is **strictly** forbidden.

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1. Choose some **strictly** increasing sequence $\{a_n\}_{n\in\mathbb{N}}$, which is bounded above.

(a) [**8**] Does the sequence $\{a_n\}_{n\in\mathbb{N}}$ converge? Justify your answer. If the answer is yes, determine also $\lim_{n\to\infty} a_n$.

(b) [**8**] Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence defined by $b_n = a_{n+1} - a_n$. Calculate the n-th partial sum S_n of the series $\sum_{k=1}^{\infty} b_k$.

(c) [**10**] Does the series $\sum_{k=1}^{\infty} b_k$ from the point (1b) converge? Justify your answer. If the answer is yes, determine also the sum of the series.

(d) [**8**] Does the answer to the question (1c) depend on the choice of $\{a_n\}_{n\in\mathbb{N}}$? If yes, find a sequence $\{\tilde{a}_n\}_{n\in\mathbb{N}}$ for which the answer changes, othwerwise justify, why the answer does not change.

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2. Answer the following tasks.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be some **even nonconstant** continuously differentiable function.

i. [**10**] Sketch the graphs of some function f with the properties above and also its derivative f'.

ii. [**8**] How are the values $f'(x)$ and $f'(-x)$ connected?

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, which satisfies $g(x) < 0 < g'(x)$ for every $x \in \mathbb{R}$.

i. [**3**] Which property does g have, since it satisfies the condition $0 < g'(x)$ for every $x \in \mathbb{R}$?

ii. [**3**] Where in $\mathbb{R}^2$ does the graph of g lie, since it satisfies the condition $g(x) < 0$ for every $x \in \mathbb{R}$?

iii. [**5**] Sketch a graph of some function g with the properties above.

iv. [**7**] Justify, that $h(x) = e^{-x} g(x)$ is an increasing function.

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3. Let $g(x,y) = x^2 + \int_0^y \sqrt{1-t^2}\,\mathrm{d}t$ be a function.

(a) [**8**] Determine the domain D_g of the function g.

(b) [**8**] Justify, that $\dfrac{\partial g(x,y)}{\partial y} = \sqrt{1-y^2}$.

(c) [**14**] Determine all **candidates** for constrained extrema of the function g restricted to the circle $x^2 + y^2 = 1$. (You do not have to classify which of the candidates are indeed constrained extrema.)

Hint. If you are not able to solve (3b), you can still solve (3c) using the rule for $\frac{\partial g(x,y)}{\partial y}$ from (3b).