Computational topology - group project

# Cards (Map Maker)

In this project you will explore topological properties of landscapes, created by random placements of special cards.

Begin by creating the cards. Take a planar shape that can be used to tile the plane: an equilateral triangle, a square, a rectangle, a hexagon, etc. Pick an even number of points on the boundary so that when the shapes are glued together, the points match up. Then pair up the points in all possible ways such that none of the lines intersect. Figure 2 shows the 14 cards we obtain if we choose two points on each of the sides of a square. Figure 3 shows the 14 cards we get if we choose four points on each of the two opposite sides of a rectangle instead.

Next, glue the cards together to form a larger shape we will call *a map*. If we use square cards, we can form a $1 \times 14$ rectangle, a $2 \times 7$ rectangle or a $7 \times 2$ rectangle. If we use the rectangular cards, we can only create a $1 \times 14$ rectangle. Figure 1 shows one possible $2 \times 7$ map from square cards.

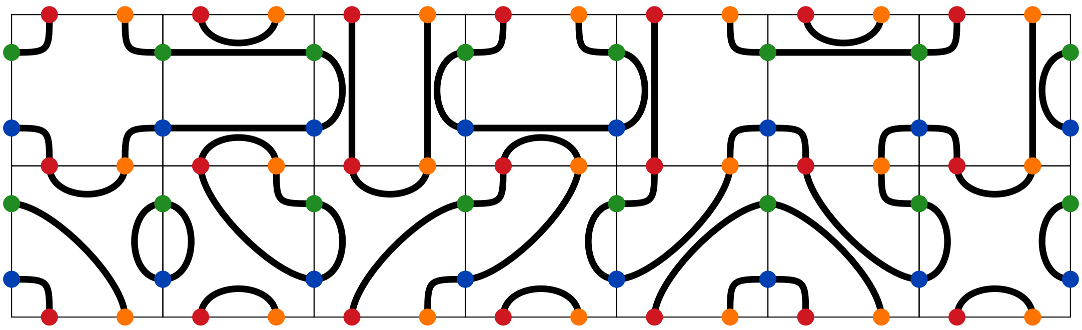

Figure 1: A $2 \times 7$ map from square cards.

In each case we can also pretend that two opposite edges of the map are glued together to form a cylinder, or that all four edges are glued in pairs to form a torus, a Klein bottle or some other 2-dimensional surface.

Once the map is created, use the tricks you have learned in class to answer the following questions:

- How many 1-dimensional components (simple closed curves) are there on the map?
- If you consider the 2-dimensional complement of these curves, how many components are there? How does the answer change if the map is a cylinder or a torus?
- Can you find the arrangement of the cards which produces the smallest possible number of the components?
- Is it clear which parts of the 2-dimensional complement are islands and which is the sea? Are there any lakes on islands?

Make sure you explain which topological tool you used to answer each of these questions. Was that the only option? If not, what were alternative ways of computing the same thing and why did you not choose those instead? Find a use for $H_1$, not only $H_0$!

The number of possible orderings of cards is very high, so it is unlikely you will be able to check them all. Do a statistical analysis of your results. Create random maps and determine the percentage of maps that have 2, 3, 4,... components. Can you estimate the number of all possible

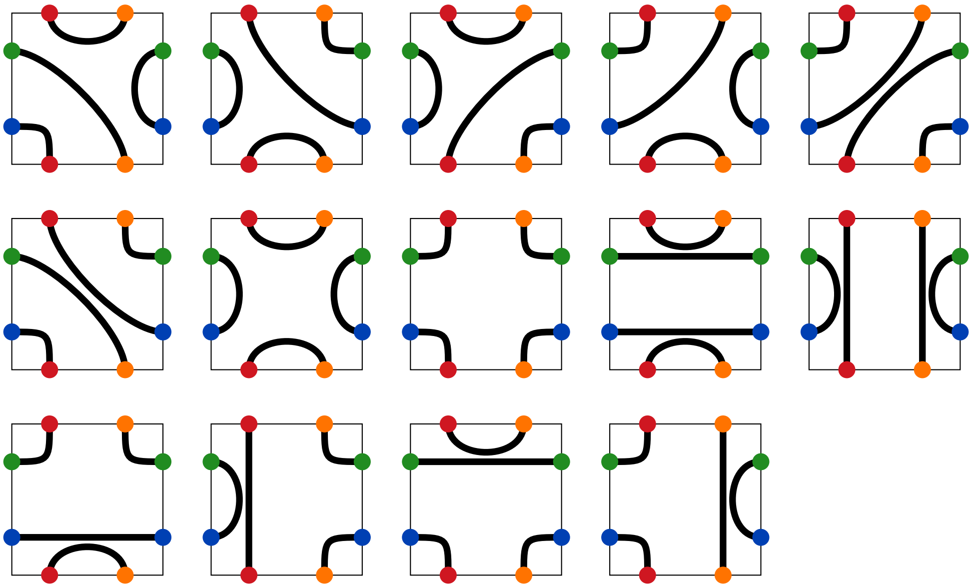

Figure 2: The cards with 2 points on each of the 4 sides of a square.

maps? Can you compute it precisely? What portion of all possible maps did you manage to analyse? What was the average number of components? Try to form some rare features by hand. Can you get them at random at all?

**Some more ideas to try**:

- Use only some of the cards, not all of them. How do the results change if you create a $2 \times 4$ rectangle instead of a $2 \times 7$ rectangle, for example? With an appropriate number of cards lined up the right way (in a square shape, for example), you can create surfaces other than the torus and the Klein bottle. Try constructing some maps on a sphere or on the projective plane. How many components do you get there?
- What happens if you allow for rotations of the cards? You can use all the cards and simply allow rotations in addition to that, or you can remove the cards that look the same after a rotation and work with a smaller subset of cards instead. See Figure 4 for an example.
- Can you create a more complicated plane-tiling pattern resembling the famous tessellations of Escher?

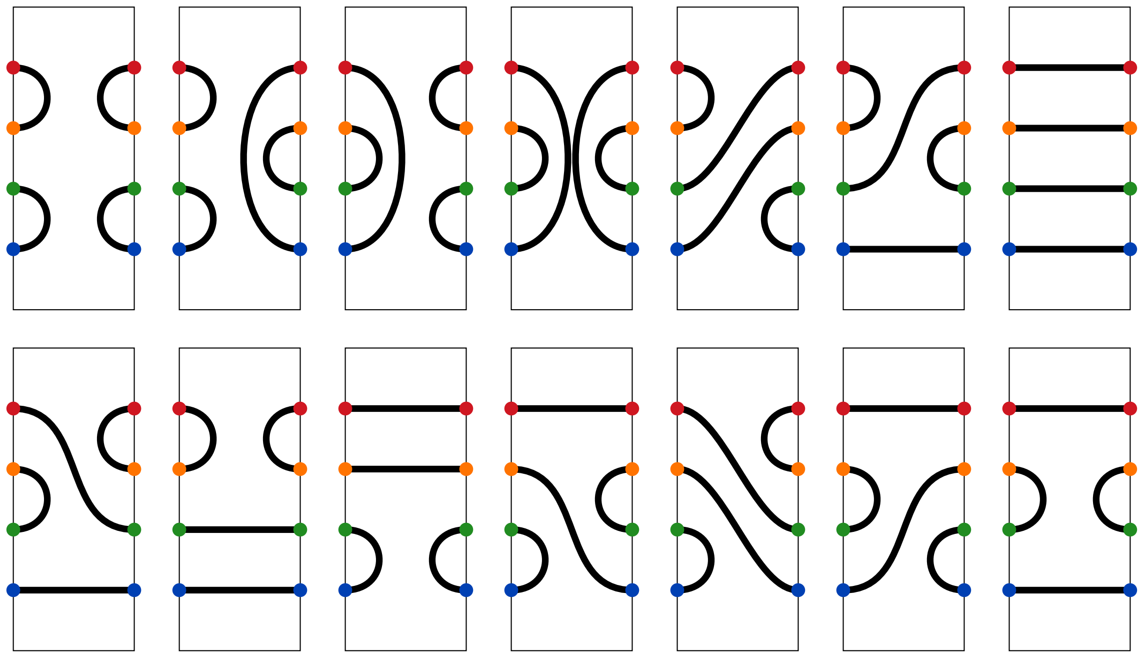

Figure 3: The cards with 4 points on each of the 2 opposite sides of a rectangle.

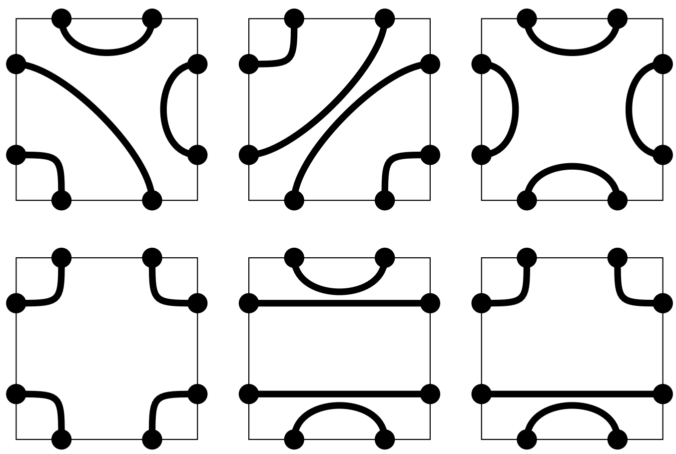

Figure 4: The cards with 2 points on each of the 4 sides of a square after removing duplicates that are the same after rotation. These cards can be rotated when combined into a map.