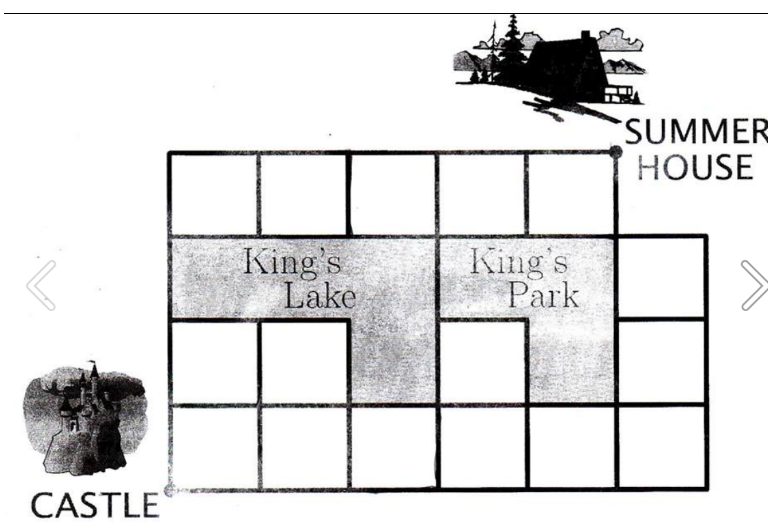
Math Problem of the Week (from Simon's Math Club):

A king loves to travel from his castle to his summer house by coach. He orders his coachmen to never go straight when he arrives at an intersection and to never travel along the same section of road twice during a trip. The coachman must turn either right or left when he comes to any intersection.

The map shows all of the roads leading from the palace to the summer house. The thick black lines indicate roads.

All roads connecting two adjacent intersections are 1 km long.

The coachman wants to take the shortest route. Determine the length of the shortest route and justify why no shorter route is possible.



(*Hint:* a construction for an upper bound) It is not difficult to find a 13km path satisfying all the requirements (it goes between the lake and the park).

If we choose a coordinate system with $(0, 0)$ at the castle and $(5, 4)$ at the house, the Manhattan distance (https://en.wikipedia.org/wiki/Taxicab_geometry) between castle/house equals 9, i.e., $5\times$ we turn East (E) and $4\times$ North (N). In this case (due to constant changing of directions) we need to start E and then N, ... : ENENENENE, however, the alternating path takes us over the lake and is therefore not possible.

Due to constant alternating between horizontal and vertical streets each castle/house path must have an odd length. On an alternating path one needs to go at least once South (S) or West (W), thus we need to add to the alternating path one N and one E/W or one W and one N/S, so the shortest path is at least $9 + 3$. Hence (EN)(ES)(EN)(EN)(WN)(EN)E is really optimal.

An alternative solution using the idea of a bottleneck:

(*Hint:* identify small number of ‘important’ points)

the optimal path has to go through one of the following points

$$(0,3), \quad (3,3) \;\text{ or }\; (5,3)$$

and has consequently length 15, 13 or (15 or 17) respectively.

Remarks. (1) Speculation: the length has the form $\;9+4k, \quad k \in \mathbb{N}$.
Due to the existence of the path of length 15 this is obviously not the case.
(2) Forbidden roads ‘spell’ a hidden message ILL.

Points to remember: (1) a construction gives an upper bound (an optimal path is close to a Manhattan optimal path), (2) special metric for lower bound, (3) invariants (parity/divisibility) for further restrictions, (4) due to a small number of possibilities we could do a computer search or (5) by hand: at each crossing write the shortest distance from/to the castle/house going L/R.