Mathematical modeling UNI, problem sessions, week 1

**Linear least squares method.** Let \( A \in \mathbb{R}^{m \times n} \) be an \( m \times n \) matrix with \( m \geq n \). Let \( b \in \mathbb{R}^m \) be a vector. How would you find the orthogonal projection of \( b \) on to the column space of \( A, C(A) \)? (Assume that the columns of \( A \) are linearly independent.)

1. We want to approximate a real function \( f \) on the interval \([a, b]\) with a polynomial. We will do this (perhaps naively) by dividing the interval \([a, b]\) with \( k + 1 \) equidistant points \( a = x_0, x_1, \ldots, x_k = b \) and then find the coefficients of the polynomial \( p(x) \) that is the best fit to the data in the table below according to the linear least squares method.

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( x_1 )</th>
<th>( \cdots )</th>
<th>( x_i )</th>
<th>( \cdots )</th>
<th>( x_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x_0) )</td>
<td>( f(x_1) )</td>
<td>( \cdots )</td>
<td>( f(x_i) )</td>
<td>( \cdots )</td>
<td>( f(x_k) )</td>
</tr>
</tbody>
</table>

(a) Let \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) be a polynomial of degree \( n \). Write the matrix \( A \) of the corresponding linear system and the right-hand side of \( b \) according to the data in the table above.

(b) Find the approximations of orders 0, 1 and 2 for the function \( f(x) = \frac{x^2}{1+x^2} \) on the interval \([-1, 1]\) using the points \( x_0 = -1, x_1 = 0 \) and \( x_2 = 1 \).

(c) Using octave approximate the function \( g(x) = \frac{1}{1+25x^2} \) on \([-1, 1]\) with polynomials of order 0, 2, \ldots, 20, dividing the interval \([-1, 1]\) with 21 equidistant points. Find the approximations for the exact data and for data with (artificially added) errors. Using the plot command plot the graphs of the original functions and all the approximations.

2. Use the linear least squares method to solve the following problem: In the plane \( \mathbb{R}^2 \) we have \( n \) transmitters in known locations \((p_1, q_1), \ldots, (p_n, q_n)\). A receiver can measure the distances from all the transmitters \( d_1, \ldots, d_n \). We want to determine the position of the receiver given the distances from the transmitters. In the ideal case the measurements are perfect and for each \( i = 1, \ldots, n \) we have an equation

\[
(x - p_i)^2 + (y - q_i)^2 = d_i^2.
\]

The solution of this system of equations then determines the unknown position \((x, y)\).

(a) The first problem is that the equations are not linear. But the difference of two consecutive equations is linear. Write down these differences to obtain a system of \( n - 1 \) linear equations.

(b) Write the matrix \( A \in \mathbb{R}^{(n-1) \times 2} \) of the system and the corresponding right-hand side \( b \in \mathbb{R}^{n-1} \). Additional problem is that the measurements are not exact which means that the system \( Ax = b \) almost surely has no solution.
(c) Find the least squares solution to $Ax = b$. Write the `octave` function $X = \text{sprejemnik}([\pi, \qi], [\di])$ that finds the position of the receiver $X(x, y)$ given the transmitter positions $(p_i, q_i)$ and distances $d_i$. (The positions $(p_i, q_i)$ are contained in an $n \times 2$ matrix and the distances $d_i$ are given in a column matrix of length $n$. The result $X$ should be row vector $X = [x, y]$.)

(d) Test the function using artificial data from the files `oddajniki.txt` and `razdalje.txt` found on `ucilnica`. You can import them into `octave` using the `load` command. Display the results.