1. The system of equations

\[
\begin{align*}
x - y + z - w &= 1 \\
x + y - z - w &= 3
\end{align*}
\]

determines a two-dimensional plane in \( \mathbb{R}^4 \). Let \( T(0, -1, -1, 2) \). Our objective is to find the point on the plane which is closest to \( T \).

(a) Write the matrix \( A \) and the right-hand side \( b \) of the system above.

(b) Evaluate \( A^+ \). (This is simple since \( A \) has full rank).

(c) Show that \( P = I - A^+ A \) is an orthogonal projection, meaning that \( P^2 = P \) and \( P^T = P \). Onto which subspace does it project?

(d) Express and compute the solution using \( A^+ \).

(e) Write the function \( p_T = \text{projekcija}(A, b, T) \), which returns the projection of the point \( T \) on to the hyperplane defined by the system \( Ax = b \).

2. SVD and image compression. A greyscale image can be represented by a matrix \( A \). (A color image can be represented using three matrices, say \( A_R, A_G \) and \( A_B \)). Using the matrices \( U, S, \) and \( V \) from the SVD decomposition we can reconstruct the matrix \( A \) by computing \( USV^T \). Moreover, we can decide that small singular values contribute very little to the image and can be ignored. Let \( S' \) be the matrix that contains the largest \( m \) singular values on the diagonal. Then \( A' = US'V^T \) can serve as an approximation to \( A \).

(a) Download the image \( \text{lena512.mat} \) and use \( A = \text{imread("lena512.mat")} \) to load it into octave/Matlab. To show the image use \( \text{imshow}(A) \).

(b) Find the SVD decomposition of \( A \).

(c) Compute the approximations for \( A \) obtained by using 10, 20, 50, 100 of the largest singular values of \( A \). Show the images and visually assess the quality of the images.

(d) How much space would we actually need to save such an approximation?

3. Linear moving average model with \( n \) delays. An unknown system is fed an input signal \( x(t) \) and returns \( y(t) \):

\[
x(t) \quad \text{unknown system} \quad y(t).
\]

Based on observations, say \( N \) measurements, we would like to predict the behaviour of our system. Suppose that the signals \( x(t) \) and \( y(t) \) are measured at times \( t = 1, 2, \ldots, N \). Based on our model, the output \( y(t) \) at time \( t \) is a linear combination of \( n \) inputs \( x(t), x(t-1), \ldots, x(t-n+1) \), ie.

\[
y(t) = h_1 x(t - n + 1) + h_2 x(t - n) + \cdots + h_{n-1} x(t - 1) + h_n x(t).
\]
Set
\[
A = \begin{bmatrix}
x(1) & x(2) & \cdots & x(n) \\
x(2) & x(3) & \cdots & x(n + 1) \\
\vdots & \vdots & \ddots & \vdots \\
x(N - n + 1) & x(N - n + 2) & \cdots & x(N)
\end{bmatrix},
\]

\[
h = \begin{bmatrix}
h_1 \\
h_2 \\
\vdots \\
h_n
\end{bmatrix}
\quad \text{and} \quad
y = \begin{bmatrix}
y(n) \\
y(n + 1) \\
\vdots \\
y(N)
\end{bmatrix}.
\]

The coefficients \(h_1, \ldots, h_n\) in the above linear combination can be obtained as the solution of \(Ah = y\) in the sense of the linear least squares method.

(a) Write an octave function \(h = \text{linmdp}(x, y, n)\) which, given input data \(x = [x(1), \ldots, x(N)]\), \(y = [y(1), \ldots, y(N)]\) finds the coefficients \(h = [h_1, \ldots, h_n]\) of the linear moving average model with \(n\) delays.

(b) Write a function \(y = \text{napoved}(x, h)\) which, given inputs \(x\) and coefficients \(h\), predicts the output \(y\) based on our model.

(c) Test your functions on the data from the files \(\text{io-ucni.txt}\) and \(\text{io-testni.txt}\). (Use the train data to determine \(h\). Test the prediction on the data from the test set.) How does the precision of prediction depend on the number of delays \(n\)?