3.2. Parametric curves

A parametric curve (or parametrized curve) in \( \mathbb{R}^m \) is a vector function \( f : I \to \mathbb{R}^m \)

\[
f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix},
\]

where \( I \subset \mathbb{R} \) is an interval (bounded or unbounded).

The independent variable (which in this case we will typically denote by \( t \)) is the parameter of the curve.

For every value \( t \in I \), \( f(t) \) represents a point in \( \mathbb{R}^m \).
\( m = 2: \) for every \( t \in I \) we get a point in the plane

\[
f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
\]

\( m = 3: \) for every \( t \in I \) we get a point in \( \mathbb{R}^3 \)

\[
f(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
\]

Example: what is the path of the valve on your bicycle wheel when you are biking along a straight road?

The wheel is modeled as a circle of radius \( a \), rolling along the real line, the valve is a fixed point on the circle.

The independent variable (the parameter) in our model will be the angle of rotation \( \theta \):

\[ x(\theta) = a\theta - a \sin \theta, \quad y(\theta) = a - a \cos \theta \]

This curve is called the \textit{cycloid}. 
More examples:

1. The function
   \[ f : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto f(t) = \begin{bmatrix} a \cos t \\ a \sin t \end{bmatrix} \]
   is the parametric circle with radius \( a \) around the origin.

2. A parametric line in \( \mathbb{R}^m \) is given by
   \[ f : \mathbb{R} \to \mathbb{R}^m, \quad t \mapsto x_0 + te, \]
   where \( x_0 \in \mathbb{R}^m \) is a point on the line and \( 0 \neq e \in \mathbb{R}^m \) is a vector pointing in the direction of the line.

3. The spiral is the parametric curve in \( \mathbb{R}^3 \) given by
   \[ f(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}, t \in \mathbb{R} \]

A curve in \( \mathbb{R}^m \) is the image of a parametric curve:
\[ C = \{ f(t), t \in I \}. \]

A given curve \( C \) can have many different parametrizations.

For example, the circle in \( \mathbb{R}^2 \) with radius \( a \) around the origin can be parametrized as

- \( f_1(t) = [a \cos t, a \sin t]^T, \ t \in [0, 2\pi] \) (the point circles around once in the positive direction starting at \( (1, 0) \)),
- \( f_2(t) = [a \sin t, a \cos t]^T, \ t \in [0, 2\pi] \) (the point circles around once in the negative direction starting at \( (1, 0) \))
- \( f_3(t) = [a \cos 2t, a \sin 2t]^T, \ t \in [0, 2\pi] \) (the point circles around twice in the positive direction starting at \( (1, 0) \))
- \( f_4(t) = [a \cos t^2, a \sin t^2]^T, \ t \in \mathbb{R} \) (the point circles around infinitely many times in the positive direction and accelerates)
Example: find the self-intersections of the curve
\[ f_1(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 - t^2 + t + 1 \end{bmatrix} \]
and its intersection with \( f_2(t) = \begin{bmatrix} t - 1 \\ 1 - t^2 \end{bmatrix} \)

- To find the self-intersections, solve the system of equations
  \[ t^2 - 1 = s^2 - 1, \quad -t^3 - t^2 + t + 1 = -s^3 - s^2 + s + 1 \]
  for the parameter values \( t \) and \( s \).
- To find the intersections of two curves, solve the system of equations
  \[ t - 1 = s^2 - 1, \quad 1 - t^2 = -s^3 - s^2 + s + 1 \]
  for the parameter values \( t \) and \( s \).

Note: two curves will typically (but not necessarily) intersect at a common intersection point at different parameter values \( t \) and \( s \).

The derivative of a vector function \( f : I \rightarrow \mathbb{R}^n \) is a column of derivatives of components:
\[
Df(t) = \begin{bmatrix}
\dot{x}_1(t) \\
\vdots \\
\dot{x}_n(t)
\end{bmatrix} = \dot{f}(t).
\]

The vector
\[
\dot{f}(t_0) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}
\]
represents the velocity vector of a point moving along the curve at \( t = t_0 \).

If \( \dot{f}(t_0) \neq 0 \) it points in the direction of the tangent to the curve at \( f(t_0) \).
If $\dot{f}(t_0) \neq 0$ the linear approximation of $f$ at $t_0$:

$$L_{t_0}(t) = f(t_0) + (t - t_0)\dot{f}(t_0)$$

is a parametrization of the tangent line to the curve at $t = t_0$.

In this case $f(t_0)$ is a **regular point** of the parametric curve and the parametric curve is **smooth** at $t = t_0$.

A curve $C \in \mathbb{R}^m$ is **smooth** at a point $x_0$ on $C$ if there exists a parametrization $f(t)$ of $C$, such that $f(t_0) = x_0$ and $\dot{f}(t_0) \neq 0$.

For example, the parametric curve $f(t) = \begin{bmatrix} \cos(t^2) \\ \sin(t^2) \end{bmatrix}$ is not smooth at the point $f(0)$ since $\dot{f}(0) = 0$.
If we introduce $u = t^2$ we obtain the usual parametrization of the circle, which is smooth at every point.

Example: the cycloid $f(t) = [a(t - \sin t), a(1 - \cos t)]^T$ is not a smooth parametric curve at points $f(2k\pi)$ where it touches the $x$ axis since $\dot{f}(2k\pi) = 0$.

The slope of the tangent line at a point $f(t)$ is $k_t = \frac{\dot{y}}{\dot{x}}$. As $t \to 2k\pi$:

$$\lim_{t \to 2k\pi} k_t = \lim_{t \to 2k\pi} \frac{a \sin t}{a(1 - \cos t)} = \lim_{t \to 2k\pi} \frac{\cos t}{\sin t} = -\infty$$

$$\lim_{t \to 2k\pi} k_t = \lim_{t \to 2k\pi} \frac{a \sin t}{a(1 - \cos t)} = \lim_{t \to 2k\pi} \frac{\cos t}{\sin t} = \infty.$$  

This shows that the curve has a sharp point at these points and a tangent does not exist.

The Cycloid is therefore not a smooth curve at its points on the $x$ axis.

(l'Hospital's rule was used to compute the limits.)
For a plane curve $f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ the tangent at a regular point $f(t_0)$ is

- the vertical line $x = x(t_0)$ if $\dot{x}(t) = 0$ and $\dot{y}(t_0) \neq 0$,
- the line $y - y(t_0) = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}(x - x(t_0))$ if $\dot{x}(t_0) \neq 0$,
- and the horizontal line $y = y(t_0)$ if $\dot{y}(t_0) = 0$ and $\dot{x}(t_0) \neq 0$.

Example: the parametric curve $f(t) = \begin{bmatrix} t^2 - 1 \\ -t^3 - t^2 + t + 1 \end{bmatrix}$

- has a self intersection at the point $(0, 0)$ which it reaches at times $t_1 = -1$ and $t_2 = 1$,
- the equations of the tangent lines at $(0, 0)$ are
  - $y = 0$ at $t = -1$, since $\dot{y}(-1) = 0$
  - $y = \frac{\dot{y}(1)}{\dot{x}(1)}x = -2x$ at $t = 1$.
- The curve has a vertical tangent at point $f(0) = (-1, 1)$ and a horizontal tangent at $f(1/3) = (-8/9, 32/27)$ and $f(1) = (0, 0)$. 

![Diagram of the parametric curve](image)
The arc length $s$ of a parametric curve $f(t), \ t \in [\alpha, \beta]$, in $\mathbb{R}^m$ is the length of the path travelled by a point on the between $t = \alpha$ in $t = \beta$.

An approximation $s_n$ for $s$ is obtained in the following way:

- subdivide the interval $[\alpha, \beta]$ into $n$ subintervals of length $\Delta t = (\beta - \alpha)/n$,
- let $t_i = \alpha + i \Delta t$ and $f(t_i), \ i = 0, \ldots, n$, be the division points and the corresponding points on the curve, respectively,
- the length of a line segments connecting two consecutive points is $\Delta s_i = \| f(t_i) - f(t_{i-1}) \| = \| \dot{f}(t_{i-1}) \| \Delta t$
- and the total length of the polygonal line connecting these points is approximated by the sum:
  \[ s_n = \sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \| \approx \sum_{i=1}^{n} \| \dot{f}(t_{i-1}) \| \Delta t. \]

If the function $f(t)$ is continuously differentiable, this converges as $n \to \infty$ to
\[ s = \lim_{n \to \infty} s_n = \int_{\alpha}^{\beta} \| \dot{f}(t) \| \, dt \]

Examples

1. The length of the path traced by the valve on the bike during one full turn, i.e. one cycle of the cycloid $f(t) = \begin{bmatrix} t - \sin t \\ 1 - 1 \cos t \end{bmatrix}$
   \[ s = \int_{0}^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} \, dt = \cdots = 8 \]

2. The arc length of the spiral $f(t) = \begin{bmatrix} a \cos t \\ a \sin t \\ bt \end{bmatrix}, 0 \leq t \leq 2\pi$, is
   \[ s = \int_{0}^{2\pi} \sqrt{a^2 \cos^2 t + a^2 \sin^2 t + b^2} \, dt = 2\pi \sqrt{a^2 + b^2}. \]

3. The circumference of the ellipse $\begin{bmatrix} a \cos t \\ b \sin t \end{bmatrix}$ is
   \[ \int_{0}^{2\pi} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} \, dt = 4a \int_{0}^{\pi/2} \sqrt{1 - e^2 \sin^2 t} \, dt = 4aE(e) \]
   where $e = \sqrt{1 - (b/a)^2}$ is its eccentricity and the function $E$ is the nonelementary elliptic integral of 2nd kind.
Exact computation of the arc length is usually tedious or impossible, so we instead use approximations:

\[ s \approx s_n = \sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \| \]

with \( n \) big enough.

That is, instead of computing the arc length of the curve, we compute the length of the linear interpolant on the division points \( f(t_i), i = 0, \ldots, n \) i.e. the polygonal line through these points.

Arc length from the initial \( t = \alpha \) to an arbitrary \( t \) is an increasing function of \( t \):

\[ s(t) = \int_{\alpha}^{t} \| \dot{f}(u) \| \, du. \]

This implies that it has an inverse function \( t(s) \), that is, the original parameter \( t \) can be expressed as a function of the arc length \( s \).

Inserting this dependence into the parametrization gives the same curve with a different parametrization: \( g(s) = f(t(s)) \).

Arc length is called the natural parameter of the curve.

If a curve \( C \) is parametrized with the natural parameter, then \( \| \dot{f}(s) \| = 1 \), so this is the unit speed parametrization.

The natural parametrization of a curve is extremely important in theory, but for practical computing it is less useful.
Example:
The standard parametrization of the circle
\[ f(t) = \begin{bmatrix} a \cos t \\ a \sin t \end{bmatrix} \]
is not the natural parametrization if \( a \neq 1 \), since
\[ \| \dot{f}(t) \| = \| \begin{bmatrix} -a \sin t \\ a \cos t \end{bmatrix} \| = a \neq 1. \]

Since \( s(t) = \int_0^t a \, dt = at \) it follows that \( t = s/a \) and the natural parametrization is
\[ g(s) = \begin{bmatrix} a \cos(s/a) \\ a \sin(s/a) \end{bmatrix}. \]

### 3.3 Plane curves

For the rest of this chapter we will study curves in \( \mathbb{R}^2 \).

**Curvature of a plane curve**

The curvature of a smooth curve at a point on the curve is the rate of change of the unit tangent vector to the curve.

If \( f(s) \) is parametrized with the natural parameter, that is \( \| \dot{f}(s) \| = 1 \), then
\[ \kappa(s) = \| \ddot{f}(s) \| \]
measures the rate of change of the *unit tangent vector*.

If \( t \) is not the natural parameter, then for a plane curve \( f(t) \)
\[ \kappa(t) = \frac{|\dot{x} \ddot{y} - \ddot{x} \dot{y}|}{\| \dot{f}(t) \|^3}. \]
Example:
The circle \( \begin{bmatrix} a \cos t \\ a \sin t \end{bmatrix} \) has natural parametrization
\[ g(s) = \begin{bmatrix} a \cos(s/a) \\ a \sin(s/a) \end{bmatrix}, \]
so
\[ \dot{g}(s) = \begin{bmatrix} -\sin(s/a) \\ \cos(s/a) \end{bmatrix} \quad \text{and} \quad \ddot{g}(s) = \begin{bmatrix} -(1/a) \cos(s/a) \\ -(1/a) \sin(s/a) \end{bmatrix} \]
so \( \kappa(s) = 1/a \).

As \( a \to \infty \), the circle goes towards a line and \( \kappa \to 0 \).

On the other hand, as \( a \to 0 \), the circle goes towards a point and \( \kappa \to \infty \).

Note: the circle is the only curve (in addition to a line) with constant curvature.

Road and railway bends, as well as roller coaster loops are designed so that the transition from the straight to the circular and back to the straight part as smooth as possible.

The transition curve from the straight part with constant curvature 0 and circular part with constant curvature \( a > 0 \) has several names: the clotoid, the Euler spiral, the Cornu spiral . . . after many people that studied it . . .

The characteristic property is that the curvature \( \kappa(s) \) is a linear function of \( s \), say
\[ \kappa(s) = ||\ddot{f}(s)|| = \alpha 2s. \]
Let us derive its equation:

Assume the parametrization is natural, that is $\dot{f}(s)$ is a unit vector, so it can be written in the form

$$\dot{f}(s) = \begin{bmatrix} \dot{x}(s) \\ \dot{y}(s) \end{bmatrix} = \begin{bmatrix} \cos \varphi(s) \\ \sin \varphi(s) \end{bmatrix}. $$

Since $\kappa(s) = 2\alpha s$ this gives

$$\kappa(s) = \sqrt{\dot{x}(s)^2 + \dot{y}(s)^2} = \dot{\varphi}(s) = 2\alpha s, \quad \varphi(s) = \alpha s^2,$$

$$\dot{x}(s) = \cos(\alpha s^2), \quad \dot{y}(s) = \sin(\alpha s^2),$$

so

$$x(s) = \int_0^s \cos(\alpha u^2) \, du, \quad y(s) = \int_0^s \sin(\alpha u^2) \, du$$

The functions $x(s)$ in $y(s)$ in the parametrization of the clotoid are nonelementary functions called the **Fresnel integrals**:

$$S(t) = \int_0^t \sin(u^2) \, du, \quad C(t) = \int_0^t \cos(u^2) \, du$$
Areas bounded by plane curves

I. Let \( f(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \) and \( \dot{x}(t) > 0 \) for \( t \in [\alpha, \beta] \)

The area of the quadrilateral bounded by the curve and the \( x \)-axis is

\[
P = \int_{x(\alpha)}^{x(\beta)} |y(x)| \, dx = \int_{\alpha}^{\beta} |y(t)| \dot{x}(t) \, dt
\]

Example: the area under one arc of the cycloid

\[
x(t) = at - a \sin t, \quad y(t) = a - a \cos t.
\]

is

\[
P = \int_{0}^{2\pi} a^2 (1 - \cos t)^2 \, dt a^2 = 3a^2 \pi.
\]

II. The area of the triangular region bounded by the curve \( f(t), t \in [\alpha, \beta] \), together with the two segments connecting the endpoint \( f(\alpha) \) in \( f(\beta) \) to the origin

is approximated by the sum of areas of small triangles subdividing the interval \([\alpha, \beta]\) into \( n \) intervals of length \( \Delta t = (\beta - \alpha)/n \).
The approximate area of a triangle with vertices \((0, 0), f(t_i), f(t_{i+1})\) is

\[
\Delta P_i = \frac{1}{2} \| f(t_{i+1}) \times f(t_i) \| = \frac{1}{2} \| (f(t_i) + \dot{f}(t_i) \Delta t) \times f(t_i) \| \\
= \frac{1}{2} \| \dot{f}(t_i) \times f(t_i) \| \Delta t = |\dot{y}(t_i)x(t_i) - \dot{x}(t_i)y(t_i)| \Delta t.
\]

The area is obtained by adding these and letting \(n \to \infty\):

\[
P = \lim_{n \to \infty} \frac{1}{2} \sum_{i=0}^{n-1} |\dot{y}(t_i)x(t_i) - \dot{x}(t_i)y(t_i)| \Delta t
\]

\[
P = \frac{1}{2} \int_{\alpha}^{\beta} |x(t)\dot{y}(t) - y(t)\dot{x}(t)| \, dt.
\]

The area bounded by
1. the asteroid ploščino astroide
   \[
x(t) = \cos^3 t, y(t) = \sin^3 t, t \in [0, 2\pi]
\]
   
   is . . .
2. the elipse \(x = a \cos t, y = b \sin t, t \in [0, 2\pi]\) is
   \[
P = \frac{4}{2} \int_{0}^{\pi/2} ab(\cos^2 t + \sin^2 t) = ab\pi.
\]
Example:

A circular tower with radius 1 is standing in the middle of a grassy lawn. A goat is tied to the tower with a rope of length 2 meters. What is the total area of grass that the goat can reach?

Let the origin of the coordinate system be in the center of the tower and the point where the goat is tied to the tower be (1, 0).

The area that can be reached by the goat is bounded by the semicircle \((x - 1)^2 + y^2 = 4, x \geq 1\) and the curves

\[
\begin{align*}
f(t) &= (\cos t, \sin t) + (2 - t)(-\sin t, \cos t), t \in [0, 2], \\
g(t) &= (\cos t, \sin t) - (2 + t)(-\sin t, \cos t), t \in [-2, 0]
\end{align*}
\]

Curves in the polar plane

The **polar coordinates** in the plane are \(r, \varphi\) where

- \(r\) is the distance to the origin, \(r \geq 0\),
- \(\varphi\), is the **polar angle**, determined up to a multiple of \(2\pi\),
- \(r = 0\) is the origin, where the polar angle is not defined,
- the ray from \(r = 0\) with \(\varphi = 0\) is the **polar axis**

If the polar axis corresponds to the positive part of the \(x\)-axis, then the connection between the polar and cartesian coordinates is:

- \(x = r \cos \varphi, y = r \sin \varphi\)
- \(r = \sqrt{x^2 + y^2}, \tan \varphi = \frac{y}{x}\)
A curve in polar coordinates is given as

\[ r = r(\varphi), \quad \varphi \in I \subset \mathbb{R}. \]

A natural parametrization is obtained by letting the parameter \( t \) be the polar angle:

\[ f(t) = \begin{bmatrix} r(t) \cos(t) \\ r(t) \sin(t) \end{bmatrix}, \quad t \in I. \]

Examples:

- \( r = 1 \) unit circle
- \( r = \varphi \) Arhimedean spiral
- \( r = \frac{1}{\varphi} \) hyperbolic spiral
- \( r = 1 - \sin \varphi \) cardioid
One more example: a butterfly

\[ r = \sin^5 \left( \frac{\varphi - \pi}{12} \right) + e^{\sin \varphi} - 2 \cos(4\varphi) \]

The logarithmic spiral is given by

\[ r(\varphi) = be^{a\varphi}, \quad f(t) = \begin{bmatrix} be^{a\varphi} \cos(\varphi) \\ be^{a\varphi} \sin(\varphi) \end{bmatrix} \]

It has the property that the angle between radius vector of a point on the curve and the tangent to the curve at that point is constant:

\[ \cos(\alpha) = \frac{a}{\sqrt{1 + a^2}} \]
The Russian alpinist Vitali Abalakov invented the *Abalakov cam* or *Abakov friend* - an alpinistic securing device that is inserted into a crack and relies on friction to prevent the climber from falling.

In 1973 the American alpinist Greg Lowe patented a cam that is shaped according to the hyperbolic spiral, so that the friction is the same for different crack sizes.

The area of a trianglura region in polar coordinates:

$$P = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\varphi$$

Example: what is the area of the region bounded by the curve

$$r(\varphi) = 1 + \frac{\cos(3\varphi)}{2}$$