Area of a triangle with vertices on three circles

You are given three circles in the plane $\mathbb{R}^2$: K_1, K_2, and K_3. A triangle ABC has vertex A on the circle K_1, vertex B on K_2 and vertex C on K_3. The objective is to find the triangle with the largest area among all such triangles ABC. Denote by (p_i, q_i) the coordinates of the centre of the circle K_i, and use $r_i > 0$ to denote the radius of the circle K_i. Write a Julia function which, for three circles given by `((pi, qi), ri)`, finds a triangle ABC with the largest possible area.

Task

1. To solve the task find the *minimum* of an appropriate function of three variables using the gradient descent method. Advice: Use the subtasks below to efficiently determine the gradient of the appropriate function. (You can, of course, suitably adjust the use of the gradient descent method...)

   (a) Let $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ be parametrizations of three (for now arbitrary) plane curves. Write down a formula for the area of a triangle with vertices on these three curves. Position vectors of vertices of this triangle are therefore $\mathbf{p}_1(t)$, $\mathbf{p}_2(u)$, and $\mathbf{p}_3(v)$, the area of this triangle is a function of three variables, denote it by $f(t, u, v)$.

   (b) Express the gradient of $-f^2$ using parametrizations $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ and their derivatives $\dot{\mathbf{p}}_1$, $\dot{\mathbf{p}}_2$, and $\dot{\mathbf{p}}_3$. Hint: Chain rule.

   (c) Write down a parametrization of a circle with radius r and centre at the point (p, q).

   (d) Use the chain rule to express $\operatorname{grad}(-f^2)$ in case $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are parametrizations of three circles.

2. Find a local minimum of $-f^2$ using the gradient descent method. Use the parameters t, u, and v, at which a minimum of $-f^2$ is attained, to evaluate the position vectors of a triangle with the largest area and use those to evaluate the actual area.

3. Write a Julia function `(tri, area) = triangle(K1, K2, K3)`, which returns the vertices A, B and C of the triangle and the area of the triangle.

   (a) `tri` is a tuple of the form `((p1, q1), (p2, q2), (p3, q3))` representing the vertices A, B and C of the triangle, the first vertex on the first circle, the second on the second and the third on the third. The coordinates of the vertices should be evaluated to **8 decimal places** (absolute error).

   (b) `area` is the area of the triangle.

   (c) `(tri, area)` is a NamedTuple with keys named `tri` and `area`.

   (d) `K` is a 3-tuple of tuples of the form `((pi, qi), ri)`, where r_i is the radius and (p_i, q_i) the centre of the circle K_i.

4. Equip the file `triangle.jl` with comments and a test function `triangle_test()`. For the test: Find a suitable configuration of circles for which you can evaluate (or guess) the solution directly (by hand).

Warning: The method just described is sensitive to the orientation of the triangle, i.e., the choice of the order of the vertices of the triangle. In general this cannot be avoided since the final solution may be a triangle with either positive or negative orientation. Nevertheless, the function `triangle` must return a triangle with the largest area. Think about how you will solve this problem. (*Hint:* Changing the order of any two vertices will change the orientation of the triangle.)

Submission

Use the online classroom to submit the following:

1. the file **triangle.jl** which should be well-commented and contain at least one test,
2. a report file **solution.pdf** which contains the necessary derivations and answers to questions.

While you can discuss solutions of the problems with your colleagues, the programs and report must be your own creation. You can use all Julia functions from lab sessions and you should assume that the `gradmet` function from lab sessions will be a part of our test environment.